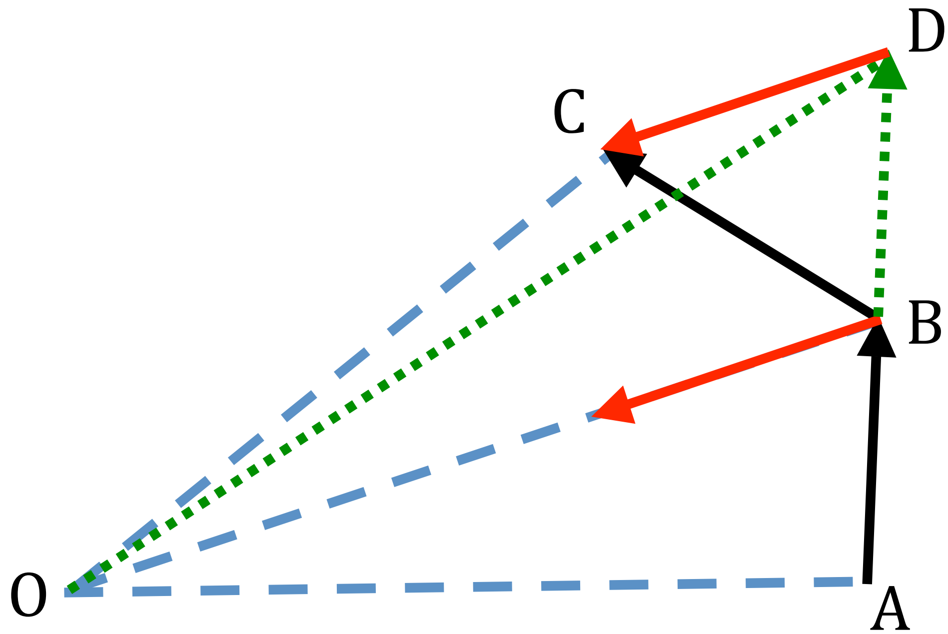
AST 120 Activity 25
Forces Toward a Fixed Center

Name	Full	Partial	None

In that last activity we saw that Newton introduced three laws of motion. The first law states that objects maintain a constant velocity unless acted on by a force. The second law states that a force causes a body to accelerate with the acceleration in the direction of the force and the magnitude of the acceleration proportional to the force and inversely proportional to the body's mass. The third law states that if one object exerts a force on another, the second objects exerts and equal and opposite force on the first. We also saw that Newton introduced the parallelogram rule to show how motions produced by a force could be combined with pre-existing (inertial) motions or motions produced by a second force. In this activity we will apply these rules to see what happens when an object is acted on by a force that always points toward a fixed point in space. The fixed point will be referred to as the *force center* and such forces are known as *central forces.*

1. To begin with, let's try to reconstruct Newton's proof that Kepler's second law works for central forces. The diagram below shows the motion of an object along the path ABC. The object is acted on by a force directed toward the fixed center at O. One problem with this situation is that the direction of the force is constantly changing (so as always to point toward O). Newton handled this by approximating the action of the force by a series of *impulses*. An impulse is like an instantaneous "kick" that imparts a change in motion to the object all at once. Newton treated the action of the central force as a series of such kicks, spaced out by equal time intervals Δt. At the end he lets $\Delta t \to 0$, so that the continuous action of the force is restored.

   In the diagram the objects starts at A. It moves inertially (at constant velocity) to B. At B it receives a kick, directed toward O and represented by the short solid arrow pointing from B toward O. The length of this arrow shows how far the object will travel *as a result of the kick only* in a time Δt. Now, if the kick had not occurred the object would have continued its inertial motion to D (where $\bar{AB} = \bar{BD}$). To find the real motion of the object we combine its inertial motion with the motion due to the kick, using Newton's parallelogram law (discussed in the last activity). So DC is parallel to BO. The resulting motion takes the object to C.



We have already shown that Kepler's second law works for inertial motion. So we know the area of triangle AOB is equal to the area of triangle BOD. To show that Kepler's law still holds when there are central forces acting on the object we must show that the area of triangle BOD is equal to the area of triangle BOC. In the space below, state an argument proving that these two triangles have equal areas. You may want to draw some additional lines on the diagram above to help illustrate your argument. Have Dr. T check your proof before you move on.

2. In the *Principia* Newton proves the converse of this theorem (ie that if the line from a fixed point to an object sweeps out equal areas in equal times then the object is subject only to forces directed toward or away from that fixed point - or else no forces at all). We know that the planets follow Kepler's second law, so based on what you just proved what does this imply about the force on the planets? How must that force be directed?

3. Newton proved that the force required to keep an object moving in a circle at uniform speed was proportional to the square of the object's speed and inversely proportional to the radius of the circle ($F \propto v^2/R$).[1] More specifically, Newton proved that the required force is $F = mv^2/R$. The proof is a bit involved and we won't go through it here (it is given as an appendix to this handout if you are interested). But to see how it works, run the `CentripetalForce` program. This shows Newton's basic method for determining the orbits of objects subject to certain forces. The object starts at some location with some velocity. In a time Δt it will undergo a movement due to its inertial motion (shown by the blue arrow). But it will also undergo a movement due to the effects of the force acting on it (shown by the red arrow). These two movements are combined using the parallelogram law to determine the overall movement of the object (shown by the magenta arrow). The simulation has been designed so that the force is proportional to the square of the object's velocity and inversely proportional to its distance from the center point. Look carefully at the simulation. The force on the object always points ______________.

   (a) in the direction the object is moving
   (b) opposite the direction the object is moving
   (c) toward the center of the circle
   (d) away form the center of the circle
   (e) none of the above

4. Does the object move in a circle? If not briefly describe the motion.

5. Now Pause the simulation. Newton was well aware that there was a major problem with his procedure. It assumes that the force on the object stays constant during the time interval Δt. But as the object moves the force will change direction (and possibly magnitude). Newton takes care of this objection by letting $\Delta t \to 0$. To see how this works, change Δt to 0.1 and change the substeps per step to 4. Click Initialize and then Clear. Now run the simulation again. Is it closer to a circular orbit this time?

6. What do you think would happen if we set Δt to 0.01? Try it, but make sure to set the number of substeps per step to 1 so it's not too slow.

[1] This result was actually first published by Christiaan Huygens in his *Horologium oscillatorium* (On the Pendulum Clock) of 1673, but Newton appears to have proved the result 10 years earlier without telling anyone.

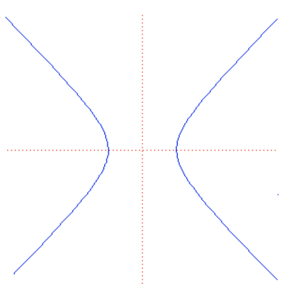
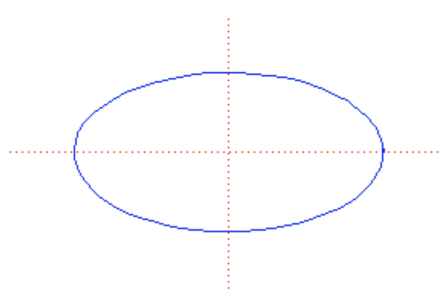
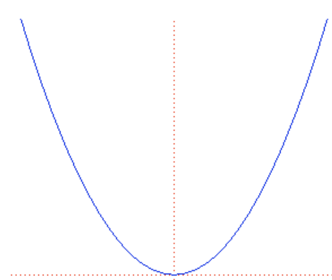
7. So we conclude that for an object undergoing uniform circular motion $F \propto v^2/R$. But the speed of the object along its orbit is simply the circumference of the circle divided by the time it takes to go around the circle. In the space below, write v as a function of R and T (the time for the object to complete one pass around the circle).

8. Substitute your above expression into v^2/R and simplify. Show your work in the space below.

9. So the central force on an object moving uniformly in a circle is proportional to ______________.
   (a) RT
   (b) R/T
   (c) R/T^2
   (d) T/R^2

10. We know that the planets move in ellipses around the sun, but their orbits are approximately circular. We also know from Kepler's third law that the squares of the periods are proportional to the cubes of the radii: $T^2 \propto R^3$. Combining this with our result above we see that (to the extent that we can treat the planets as circles) the force on the planets is proportional to ______________.
    (a) R^2
    (b) R
    (c) $1/R$
    (d) $1/R^2$

11. Now all this seems to work for circular orbits. But Newton knew that the planets moved in ellipses with the Sun at one focus (Kepler's first law). So he had to show that an inverse-square force directed toward the Sun would produce an elliptical (or circular - a circle is basically a form of an ellipse) orbit. Actually, in the *Principia* he proves the converse of this (ie that an elliptical orbit must be produced by an inverse-square force directed toward one focus). He also shows that a parabolic path and a hyperbolic path imply an inverse-square force directed toward one focus. The images below show the *conic sections*: ellipse, parabola, and hyperbola. Write the name of each one under the appropriate image.

12. Now run the `InverseSquare` program. This simulation shows the path traveled by an object (magenta dot) subject to an inverse-square force directed toward a fixed center (orange dot). The blue arrow shows the direction and magnitude of the object's instantaneous velocity, while the red arrow shows the direction and magnitude of the object's instantaneous acceleration. The orbit is calculated using the same approach as in the `CentripetalForce` program above (ie inertial motion over a short time interval plus accelerated motion over the same time interval), but the simulation uses some fancier mathematics to accurately go to the limit $\Delta t \to 0$.[2] The orbit produced by the simulation when you first run it is ______________.
    (a) an hyperbola
    (b) a circle
    (c) an ellipse
    (d) a parabola
13. Now Pause the simulation and change the initial speed to 2.5. Click Initialize and then Clear. Play the simulation again. This time the orbit is ______________.
    (a) an hyperbola
    (b) a circle
    (c) an ellipse
    (d) a parabola
14. Pause the simulation again, set the initial speed to 5. Click Initialize and then Clear. Play the simulation again. This time the orbit is ______________.
    (a) an hyperbola
    (b) a circle
    (c) an ellipse
    (d) a parabola
15. Pause the simulation again, set the initial speed to 6. Click Initialize and then Clear. Play the simulation again. This time the orbit is ______________.
    (a) an hyperbola
    (b) a circle
    (c) an ellipse
    (d) a parabola

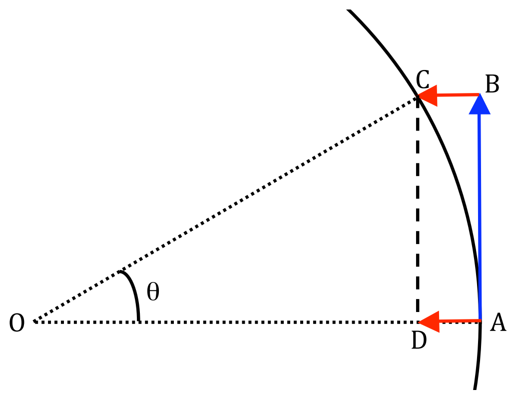
So Newton showed that to satisfy Kepler's second law the force on the planets must be directed toward the Sun. He also showed that to satisfy Kepler's first law the force must be inversely proportional to the square of the distance from the Sun to the planet. He also showed that such an inverse-square force would lead to Kepler's third law.[3] So with his three laws of motion (plus the parallelogram rule) and one inverse-square force Newton has reproduced all of Keplerian astronomy, which as we have seen was more precise than any astronomical system ever before devised. But Newton is not done yet. His new physics is not *just* a physics of celestial motions. It can explain a lot of things here on the Earth as well...

[2] The fancy mathematics is calculus, with which you may be familiar. Newton invented calculus to deal with problems like this in which he needed to take the $\Delta t \to 0$ limit, as well as other problems like how to determine the total force on a solid body (rather than a point). As with so many other things, Newton didn't publish his work. As a result, Leibniz published a slightly different version of the calculus (which he had developed independently) before Newton's work was made public.

[3] We showed that it would lead to Kepler's third law for *circular* orbits, but Newton extended this result to elliptical orbits.

Force is proportional to v^2/r for uniform circular motion

The goal is to determine the force necessary to keep an object moving in a circular path at constant speed. The diagram below shows an object moving on a circular path. At A it is moving directly upward (tangent to the circle) with speed v. The object's inertial motion carries it to point B in a time Δt. However, while it is at A the object is also acted on by a force directed toward the center of the circle at O. Although this force will change its direction as the object moves, we will treat it as though it remains constant in magnitude and direction during the time Δt. This approximation will be justified when we let $\Delta t \to 0$. As a result of this force the object attains a motion shown by the arrow BC. The total motion of the object is the combination of AB and BC. We want to determine the force that is needed to ensure that C is on the circle.



For inertial motion the distance is just the velocity times the time, so $\bar{AB} = v\Delta t$. For motion with constant acceleration the distance traveled by the object is equal to one half the acceleration times the square of the time. So $\bar{BC} = a\Delta t^2/2$. But from trigonometry we know that $\bar{OD} = R\cos\theta$, where R is the radius of the circle. Therefore we see that $\bar{BC} = R - R\cos\theta$. But for small angles, $\cos\theta \approx 1 - \theta^2/2$ where θ is measured in radians. So we see that $\bar{BC} \approx R\theta^2/2$.[4]

Now the angle θ (in radians) is just the distance the object has moved along the circle divided by the radius of the circle, or

$$\theta = \frac{\sqrt{(v\Delta t)^2 + (a\Delta t^2/2)^2}}{R}$$

but when we let $\Delta t \to 0$ the a term will be MUCH smaller than the v term so we can safely ignore it and we find

$$\theta = \frac{v\Delta t}{R}.$$

Substituting this back into our previous result ($\bar{BC} = R\theta^2/2$), we find

$$\bar{BC} = R\left(\frac{v\Delta t}{2R}\right)^2 = \frac{v^2\Delta t^2}{2R}.$$

But we already had that $\bar{BC} = a\Delta t^2/2$, so therefore

$$\frac{a\Delta t^2}{2} = \frac{v^2\Delta t^2}{2R}$$

and finally

$$a = \frac{v^2}{R}.$$

Since $a = F/m$ we have that

$$F = \frac{mv^2}{R}.$$

All our approximations are justified by taking $\Delta t \to 0$.

[4]Newton didn't use $\cos\theta \approx 1 - \theta^2/2$. Instead he used a quantity called the *versed sine* which is defined as $1 - \cos\theta$. He showed (in Corollary 1 of Lemma 11) that the versed sine is proportional to the square of the sine as $\theta \to 0$. The result is the same.