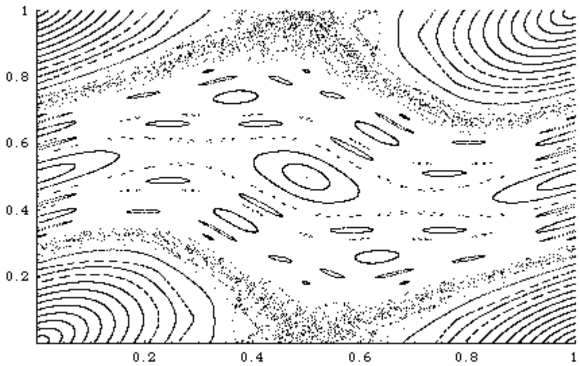
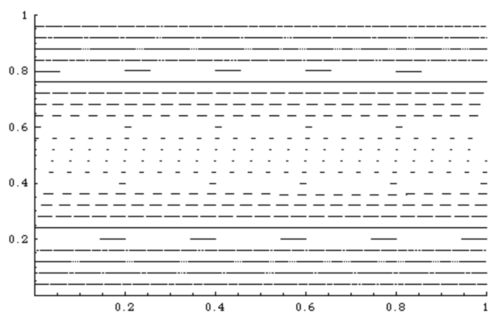
Chaos in Conservative Systems



Integrable Systems

- We have seen that integrable systems can be described in terms of action-angle variables.
- In terms of these variables, the motion of an integrable system can be thought of as the winding of a trajectory on an N-torus, where N is the number of degrees of freedom of the system.
- The torus associated with a particular set of action variables is called an **invariant torus** (or **KAM torus**).

KAM Tori (Standard Map for K = 0.001)



Definition of Chaos

- Chaotic systems have sensitive dependence on initial conditions. This means that changing the initial conditions slightly results in a dramatic change in the behavior.
- Note that integrable systems (winding on their invariant tori) cannot have sensitive dependence on initial conditions, so they are never chaotic.
- It is speculated that all non-integrable systems have some chaotic motion, but this is not certain.

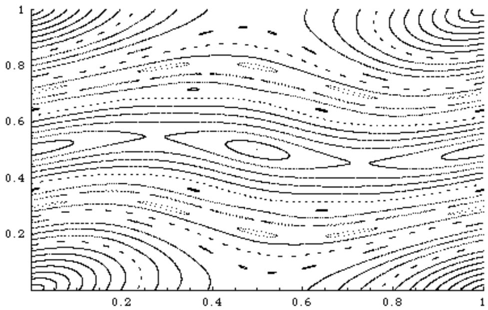
How to Make a Chaotic Sytem From and Integrable One

- Suppose we have an integrable system with Hamiltonian H_0.
- We can perturb the system by adding a nonlinear interaction term: $H = H_0 + \varepsilon V$
- Think of two uncoupled harmonic oscillators (integrable), and then add a non-linear coupling between the oscillators.
- The parameter ε determines the strength of the nonlinearity. As it is increased the system may become chaotic.

Nonlinear Resonance

- As we have seen, integrable motion can be described by angle variables that rotate at certain frequencies.
- When these different motions become coupled by a nonlinear perturbation, their frequencies may be altered.
- If the frequencies of motion in different dimensions are equal, or at least rationally related (i.e. the **winding number** is rational), then a **nonlinear resonance** may result in which the two motions become **phase locked** (creating a periodic orbit in phase space). Nearby trajectories wind around this periodic orbit. The location of a nonlinear resonance can change as the perturbation is increased.

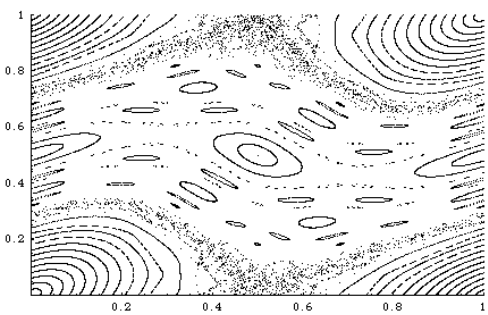
Nonlinear Resonances (Standard Map for K = 0.6)



Resonance Overlap

- The size of these nonlinear resonances depends upon the strength of the nonlinearity, ε.
- As ε is increased these resonances grow larger, and eventually may overlap with one another.
- When resonances overlap, trajectories cannot "decide" which resonance to follow. This results in the breaking of KAM tori and localized chaotic motion in the vicinity of the broken torus.
- The resulting phase space will show a mix of regular and chaotic motion. This is called a mixed phase space.

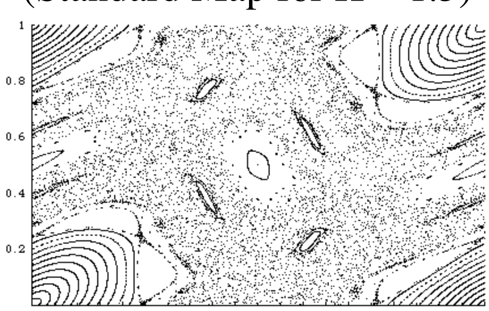
Broken KAM Tori (Standard Map for K = 0.95



Global Chaos

- If the nonlinearity parameter is increased still further, more KAM tori will be broken.
- Eventually the last KAM torus will be broken and there will be chaotic motion throughout the phase space. Although some regions of regular motion may remain, they will be **islands in a chaotic sea.**

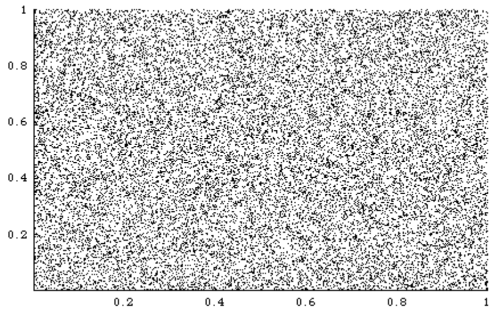
Global Chaos (Standard Map for K = 1.5)



Hard Chaos

- As the nonlinearity parameter becomes very large the remaining islands of regular motion may be destroyed, leaving the entire phase space chaotic.
- In this case the phase space is no longer mixed, but is instead completely chaotic. This is called **hard chaos**.

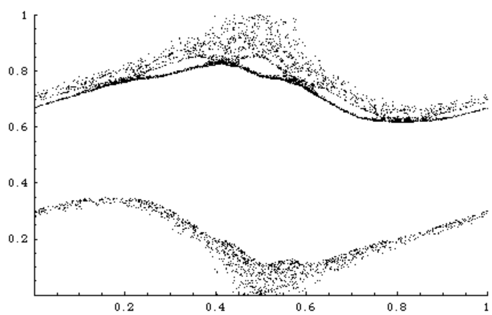
Hard Chaos
(Standard Map for K = 7)



Chaotic Trajectories
(Diffusion Through Phase Space)

- A trajectory that starts out in a chaotic region of phase space will diffuse (wander) through the chaotic region.
- If the chaos is localized, then the trajectory will be confined to that region.
- If the chaos is global, then the trajectory is not confined.

Mixed Phase Space
(Chaotic Trajectory for K = 0.95)



Global Chaos
(Chaotic Trajectory for K = 1.5)