# Poincare Sections

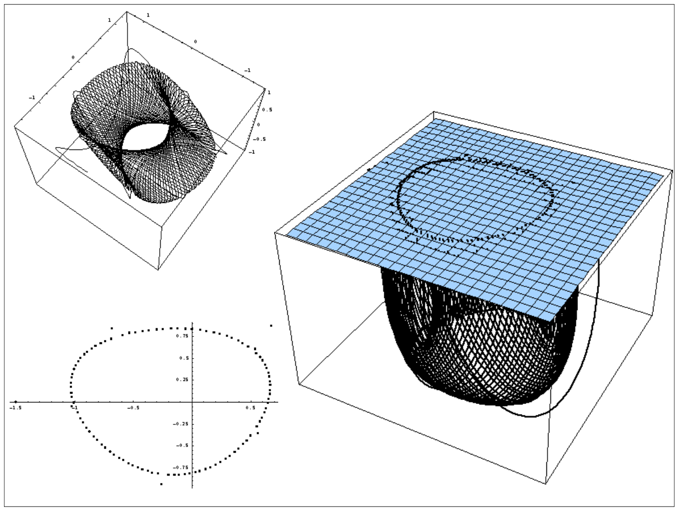
Visualizing Chaos

## Trajectories Are Hard to Picture

- For a system with 2 degrees of freedom, the motion takes place in a 4 dimensional phase space (2 coordinates, 2 momenta).
- We cannot visualize 4-D graphs, or plot them on a computer.
- This gets even worse for more degrees of freedom.

## Poincaré Sections

- Poincaré came up with a way to visualize the motion of a 2 degree of freedom system.
- Suppose you use action-angle variables of the unperturbed system ($\theta_1$, $I_1$, $\theta_2$, $I_2$) to describe the motion of the perturbed system .
- All four of these quantities may change as the system moves (actions are not constant in the perturbed system).
- Poincaré Section: Whenever $\theta_2 = 0$ and is increasing, plot the values of ($\theta_1$, $I_1$) at that time. This produces a 2-D plot that can help you visualize the dynamics of the system.



## Poincaré Sections are Maps

- We can think of Poincaré sections as maps that map a particular point ($\theta_{1,n}$, $I_{1,n}$) on the plot to the next point ($\theta_{1,n+1}$, $I_{1,n+1}$).
- This map is deterministic: since energy is conserved, if we know 1 coordinate and 2 momenta (remember $I_2 = 0$ for all points on the plot) we can find the other coordinate. Once we know all the initial conditions we can solve for the behavior of the system (numerically if necessary).

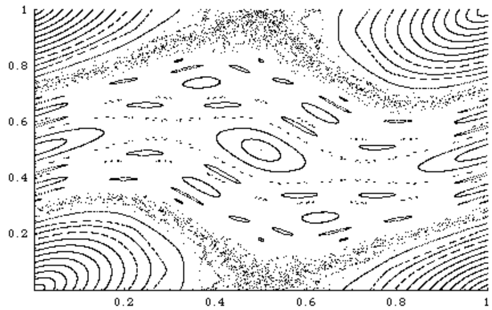
## The Standard Map

- The standard map, which models the kicked rotor system, is given by:

  $r_{n+1} = r_n - (K/2\pi) \sin(2\pi\theta_n)$, Mod 1

  $\theta_{n+1} = \theta_n + r_{n+1}$, Mod 1
- Iterating this map produces Poincaré sections.

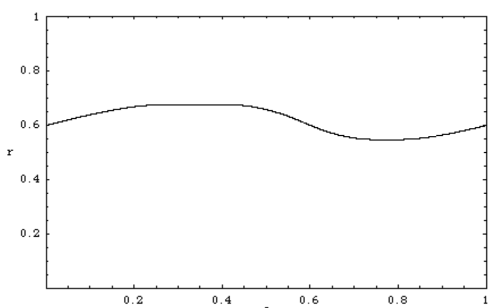
# Standard Map for K=0.95



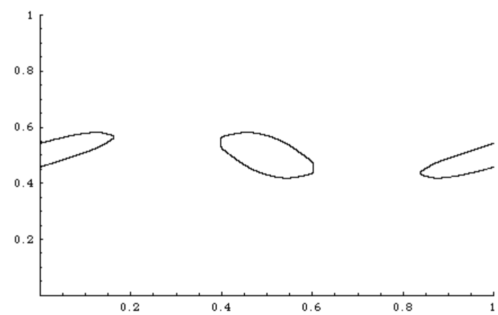

# Types of Trajectories

- Depending on the initial conditions you choose the trajectory may be in a KAM torus, a resonance, or may be chaotic.
- You can use the Poincaré Section to visualize the dynamics of these trajectories.

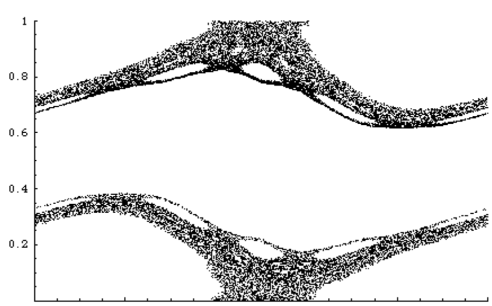
# KAM Trajectory (K = 0.8)




# Resonance Trajectory (K = 0.95)



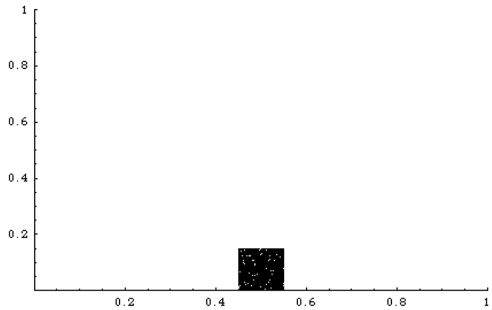

# Chaotic Trajectory (K = 0.95)



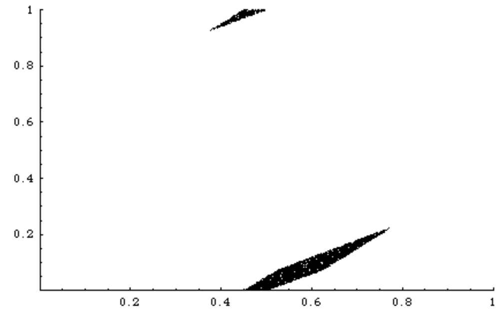

# Poincaré Sections Preserve Area

- Imagine starting with a bunch of points whose initial conditions are distributed evenly over some small region of phase space.
- If we apply the Poincaré map to all of these points, the resulting distribution of points will occupy the same area of phase space (if our system is conservative).
- Generally, the distribution spreads out in one direction and compresses in another.
- If the motion is chaotic, the distribution can become highly convoluted.

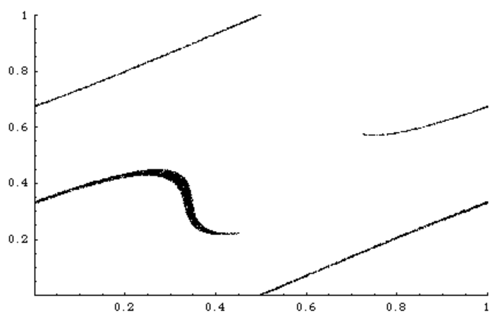
## Initial Conditions



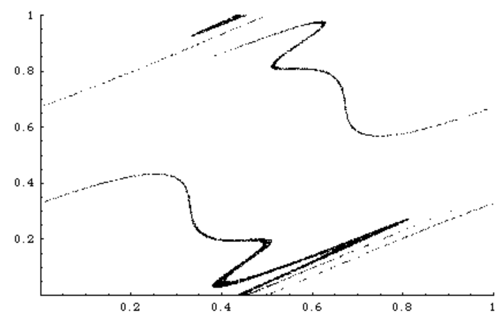

## After One Iteration



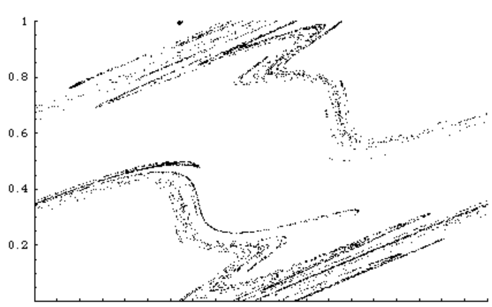

## After 3 Iterations




## After 5 Iterations




## After 10 Iterations




## Condition for Area Preservation

- For a phase space map to preserve area, the determinant of its Jacobian must equal 1.
- If the phase space coordinates are θ and I, then the Jacobian is defined to be the matrix of partial derivatives of the coordinates and momenta after n+1 iterations with respect to the coordinates and momenta after n iterations.

$$J = \begin{pmatrix} \frac{\partial \theta_{n+1}}{\partial \theta_n} & \frac{\partial \theta_{n+1}}{\partial I_n} \\ \frac{\partial I_{n+1}}{\partial \theta_n} & \frac{\partial I_{n+1}}{\partial I_n} \end{pmatrix}$$