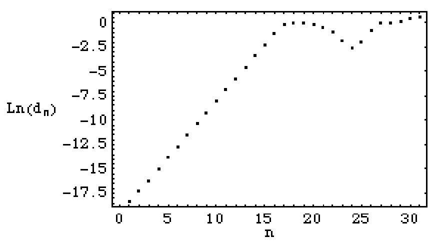
Lyapunov Exponents

Quantifying Chaotic Motion

Divergence of Trajectories Close to a Hyperbolic Point

- Consider a trajectory on the unstable manifold (H_-) of a hyperbolic point (H).
- Each time we apply the map this point moves a factor of $|\alpha|$ farther from H, where α is the eigenvalue of the stability matrix of H with magnitude greater than one.
- After N iterations of the map, the distance of the point from H is $d_N = d_0|\alpha|^N$, where d_0 is the initial distance of the point from H.
- Clearly the trajectory diverges exponentially from H as N is increased.



- This plot shows the natural log of the distance between two initially close trajectories as a function of the number of map iterations.
- The initial points for the two trajectories were (0.5,0) and (0.50000001,0).
- The straight line with positive slope for n<17 shows exponential divergence. This divergence levels off when the trajectories reach the phase space boundaries.

Lyapunov Exponent for a Hyperbolic Point

- We define the Lyapunov exponent to be $\lambda=\ln|\alpha|$, so that $d_N = d_0 e^{\lambda N}$.
- Note that since $|\alpha|$ is greater than one, λ is a positive number.
- We can also calculate a Lyapunov exponent for the other eigenvalue ($1/|\alpha|$). This Lyapunov exponent will be the negative of the first Lyapunov exponent.
- For Hamiltonian (area-preserving) systems, the sum of the two Lyapunov exponents must be zero.

Lyapunov Exponents for a Stable Fixed Point

- For a stable fixed point the eigenvalues of the stability matrix are of the form $e^{\pm i\beta}$. So the Lyapunov exponents are zero.
- So non-zero Lyapunov exponents (one positive, one negative) imply that the orbit is unstable. Zero Lyapunov exponents imply that the orbit is stable.

Lyapunov Exponents for the Standard Map

- Calculate the Lyapunov exponents for the fixed point of the Standard Map at (0.5,0.5) for K=1.5 and K=7.

Lyapunov Exponents for a Periodic Trajectory

- To calculate the Lyapunov exponent for, say, a period-3 orbit you must first find the 3 points in the orbit: z_0, z_1, z_2.
- The calculate the tangent map for each one of these points: P_0, P_1, P_2.
- Find the eigenvalue α of (P_2 P_1 P_0). The Lyapunov exponent is $\lambda = \ln(\alpha)/3$.
- In general λ = ln(Eigenvalue of $P_N \ldots P_0$)/ N.

Lyapunov Exponents for Any Trajectory

- We don't have to be near a fixed point to find a Lyapunov exponent. The Lyapunov exponent measures the exponential divergence of nearby trajectories using **any** trajectory as the origin.
- Consider a (non-periodic) point $z_0 = (\theta_0, I_0)$.
- We can define a tangent map for that point which we will call P_0.
- After one iteration of the map, the point will be at $z_1 = z_0 + P_0\, z_0$.

- We must then define a new tangent map for the point z_1, which we will call P_1.
- After two iterations of the map the point is at $z_2 = z_0 + P_1\, z_1 = z_0 + P_1\, P_0\, z_0$.
- After N iterations the point will be at $z_N = z_0 + (P_N \ldots P_1\, P_0)\, z_0$.
- The Lyapunov exponents of the trajectory can be found by taking the limit as N approaches infinity of the natural log of the eigenvalues of ($P_N \ldots P_1$ P_0), divided by N.
- Nonzero values indicate exponentially diverging trajectories (chaos). Zero values indicate stable motion.

Measure of Chaos

- The largest Lyapunov exponent serves as a measure of chaos in a region of phase space.
- If a trajectory has a Lyapunov exponent λ, nearby trajectories diverge from it as $e^{\lambda N}$. This is true in the limit of large N even if the point is not on the unstable manifold of a fixed point.
- This means that for larger λ, nearby trajectories diverge more rapidly and thus the region of phase space near the base trajectory is "more chaotic".

Numerical Determination of λ

- Finding the Lyapunov exponents for a non-periodic trajectory can be difficult.
- The tangent maps must be accurately calculated for each point on the trajectory.
- N must be large enough that the eigenvalues of ($P_N \ldots P_1$ P_0) divided by N converge to a single value.
- This can be done numerically, but it is not an easy task.