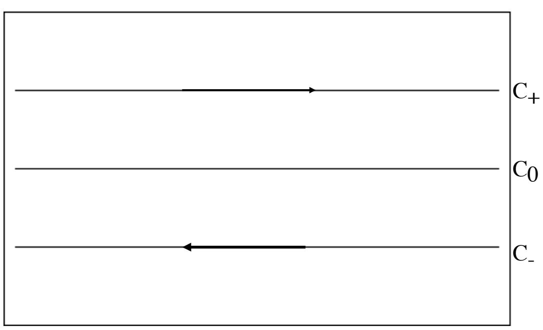
The Poincare-Birkhoff Theorem

How Do KAM Tori Break Up?

Winding Directions

- For an integrable system, the phase space consists of a series of n-tori ordered by winding number.
- Consider trajectories on three nearby tori (which we will call C_+, C_0, and C_-):
  - C_+ has the largest winding number of the three
  - C_- has the smallest winding number of the three
- Switch to a frame that winds with C_0, so C_+ moves to the right in phase space and C_- moves to the left.

Integrable Picture



Action of the Poincare Map

- Let C_0 be a trajectory with a rational winding number r/s.
- Apply the Poincare map (T) **s** times (or the map T^s once) to any point on C_0 and you get back to the same point (you have gone around the torus r times).
- On the other hand T^s causes rightward motion on C_+ and leftward motion on C_-.
- So examining the dynamics of T^s is equvalent to switching to a frame that winds with C_0.

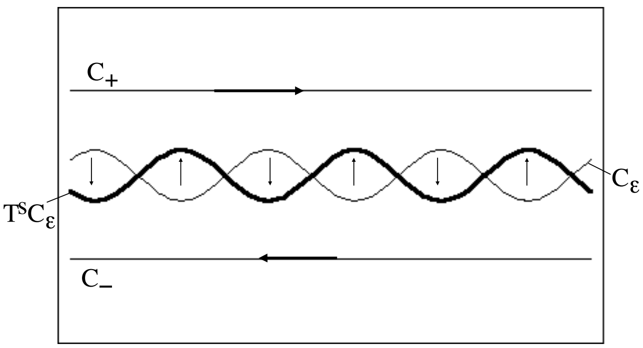
Perturbed System

- When the perturbation is turned on the KAM tori become distorted.
- For small perturbation we will still have C_+ winding right and C_- winding left under the action of T^s.
- This means that as we move through the KAM tori there will be some point between C_+ and C_- where there is no winding. In other words the angle doesn't change, although the action may.
- We will call the set of all points for which this is true C_ε. Note that this no longer represents an actual trajectory.

Action of T^s on C_ε

- If we apply the map T^s to the set of points on C_ε they can change action, but they cannot change angle.
- The result of applying the map gives a new set of points which we will call T^sC_ε.
- Because the map is area-preserving (as discussed before) we know that the area under (inside) T^sC_ε and C_ε in phase space must be the same.

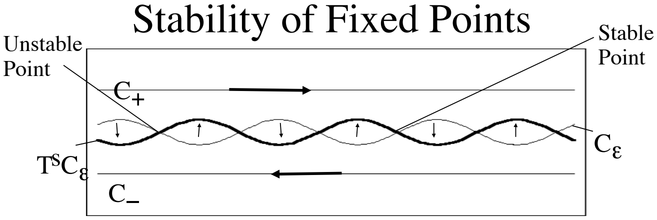
T^sC_ε and C_ε



Poincare-Birkhoff Theorem

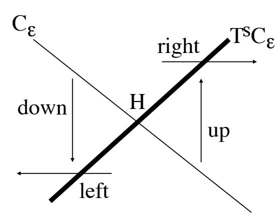
- The points where T^sC_ε and C_ε intersect are fixed points of the map T^s (or periodic points of T with period s).
- Since the area under the two curves must be the same, the curves must intersect (except in the trivial case where T^s is the identity).
- There must be an even number of (non-grazing) fixed points. The pathological case of the two curves grazing each other does produce a fixed point but we will ignore this possibility for now as it will not affect our basic result.
- We will see that half of the (non-grazing) fixed points are stable (elliptic) and half are unstable (hyperbolic).

Stability of Fixed Points



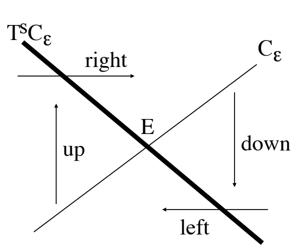
- Points on T^sC_ε that are below C_ε will move left.
- Points that are above C_ε will move right.
- This leads to an alternating series of stable and unstable fixed points.

Unstable Fixed Point



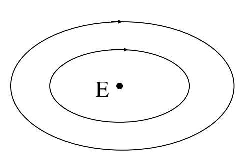
- The map T^s carries a point from C_ε to a point on T^sC_ε
- Continuing to apply the map causes the point to move away from the fixed point (H).
- This makes the fixed point unstable.

Stable Fixed Point



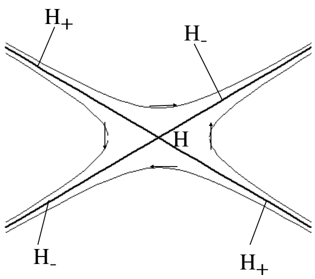
- The map T^s carries a point on C_ε to a point on T^sC_ε.
- Continuing to apply the map causes the point to wind around the fixed point (E), making that fixed point stable.

Motion Near a Stable Fixed Point



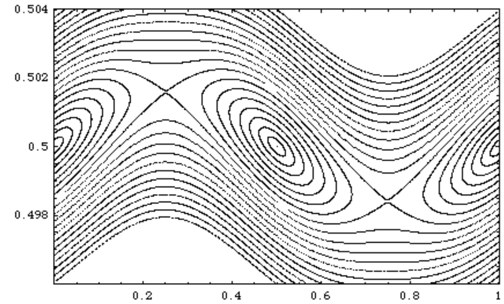
- Trajectories near a stable fixed point move in ellipses centered on the fixed point.
- This is why stable fixed points are called elliptic.

Motion Near an Unstable Fixed Point



- Trajectories near an unstable fixed point move toward the fixed point along the stable manifold (H_+) but away along the unstable manifold (H_-)
- The resulting path is a branch of a hyperbola, which is why unstable fixed points are called hyperbolic.

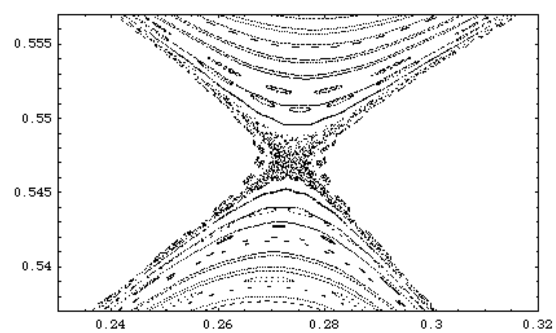
Elliptic and Hyperbolic Points Around N=2 Resonance (K=0.2)



Hyperbolic Points and Chaos

- In nonintegrable systems there will be a thin region of chaos around the stable and unstable manifolds of a hyperbolic fixed point.
- There may also be secondary (or “daughter”) resonances at the edges of this thin chaotic region. These have their own hyperbolic fixed points.
- As the perturbation is increased the chaotic regions around the manifolds of the hyperbolic points combine and grow larger.

Hyperbolic Point, K=0.6



Hyperbolic Point, K=0.7