The Tangent Map and the Stability Matrix

Tracking nearby trajectories

What happens to nearby trajectories?

- Consider a map (T) defined by
  $\theta_{i+1} = f(\theta_i, I_i)$ and $I_{i+1} = g(\theta_i, I_i)$
- The point (θ_0, I_0) is mapped to the point $(\theta_1, I_1) = (f(\theta_0, I_0), g(\theta_0, I_0))$.
- What happens to the nearby (infinitesimally close) point $(\theta_0+\delta\theta_0, I_0+\delta I_0)$? It is mapped to a point that is near (θ_1, I_1), which we will call $(\theta_1+\delta\theta_1, I_1+\delta I_1) = (f(\theta_0+\delta\theta_0, I_0+\delta I_0), g(\theta_0+\delta\theta_0, I_0+\delta I_0))$.

Taylor Series Expansion

- Using a two-variable Taylor expansion of f and g about (θ_1, I_1) we find that to first order
  $\delta\theta_1 = (\partial f/\partial\theta)|_{(\theta 0, I0)}\ \delta\theta_0 + (\partial f/\partial I)|_{(\theta 0, I0)}\ \delta I_0$
  $\delta I_1 = (\partial g/\partial\theta)|_{(\theta 0, I0)}\ \delta\theta_0 + (\partial g/\partial I)|_{(\theta 0, I0)}\ \delta I_0$
- We can represent this as a matrix equation:
  $\delta\mathbf{z_1} = \mathbf{P}\,\delta\mathbf{z_0}$ where $\delta\mathbf{z} = (\delta\theta, \delta I)$ and $\mathbf{P}$ is the matrix of partial derivatives as shown above
- The matrix **P** is called the **tangent map** or **monodromy matrix** of the Poincare map. If the initial conditions are those of a fixed point it is called the **stability matrix** of the fixed point.
- Note that **P** is just the **Jacobian** of the map, evaluated at a certain point.

Standard Map

- Find the tangent map for the Standard Map.
- Evaluate the tangent map for the initial condition (0, 0). Leave K as an undetermined variable.
- Now evaluate the matrix for K = 1.5. Use this to estimate the result of applying the map to (0.001, 0.001).
- Also, evaluate the matrix for K = 7. Estimate the result of mapping (0.001, 0.001).
- We will make use of these results when we discuss Lyapunov exponents.

Tangent Map for a Continuous System

- The procedure is more complicated for a continuously evolving system.
- In that case, the equations of motion are differential equations. We must numerically integrate a set of linearized differential equations to find the tangent map for the n x n system.
- The n x n tangent map matrix must then be reduced to a 2 x 2 stability matrix for the Poincare map.
- This is difficult, but can be done numerically.

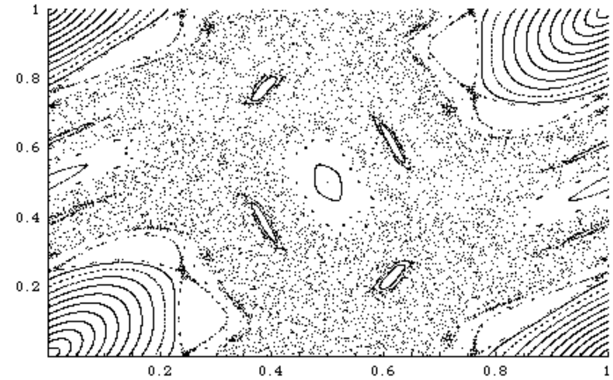
Trace of the Stability Matrix

- The nature of a fixed point can be determined by the trace of its stability matrix.
- If -2 < Tr **P** < 2: the fixed point is stable.
- If Tr **P** > 2: the fixed point is unstable and is called a “saddle” hyperbolic point. Nearby trajectories move away along one side of the unstable manifold (H_-).
- If Tr P < -2: the fixed point is unstable and is called a “reflection” hyperbolic point. Nearby trajectories move away along the unstable manifold but hop from one side to the other.

Standard Map Again

- Determine the stability of the fixed point of the Standard Map at (0, 0) for K = 1.5, and K = 7.
- Compare your results to the phase space plots for these parameters.

K = 1.5



K = 7