The Origins of Chaos

Homoclinic and Heteroclinic Tangles

H_+ and H_-

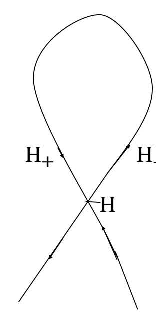
- If a point is on the stable (H_+) or unstable (H_-) manifold of a hyperbolic point (H), it will always remain on that manifold under iterations of the map.
- The stability matrix of a hyperbolic point has eigenvalues that are real and reciprocals of each other (α and $1/\alpha$).
- Close to the hyperbolic point, H_+ runs along the direction of the eigenvector of the stability matrix with eigenvalue less than one. H_- runs along the eigenvector with eigenvalue greater than one.
- As they get farther from the hyperbolic point the manifolds will no longer be straight lines.

Unstable Fixed Points in Integrable Systems

- Not all unstable fixed points necessarily lead to chaos. For instance, the pendulum is integrable but has an unstable fixed point.
- In integrable systems, H_- and H_+ join together smoothly to form a single, simple structure called the separatrix (as in the pendulum).

Connecting H_- and H_+

- Points on H_- move away from H, while points on H_+ move toward H.
- If the system is integrable these manifolds connect smoothly, points will initially move away from H along H_- and then move back toward H along H_+.
- The unstable manifold of one hyperbolic point can also connect to the stable manifold of another.

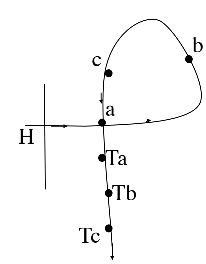


Nonintegrable Systems

- In nonintegrable systems H_+ and H_- don't connect. This is the essential difference between integrable (nonchaotic) and nonintegrable (chaotic) systems.
- If H_+ crosses H_- once, it must cross an infinite number of times (shown by Poincarè).
- Neither H_+ nor H_- can cross itself (also Poincarè).
- The area enclosed by the curves between successive crossings of H_+ and H_- must always be the same because the map is area-preserving.
- The combination of these elements leads to a very complicated structure called a tangle.

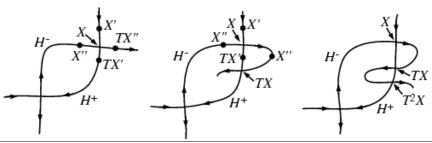
No Crossing

- Consider an unstable manifold H_- that crosses itself. It must make a loop as shown.
- The points a, b, and c are mapped to the points Ta, Tb, and Tc by one iteration of the map.
- Although c is closest to a, Tb is closest to Ta. This contradicts the continuity of the map (neighboring points are mapped to neighboring points), so this situation cannot occur.



Infinite Intersections

- A point on H_+ is always mapped to another point on H_+. The same is true for H_-.
- So a point where H_+ and H_- intersect must be mapped to another intersection point.
- So if there is one intersection point (and it isn't on a periodic orbit) then there must be an infinite number of intersection points.
- Periodic orbits can't be on H_+ or H_- because they don't move consistently away or toward H.
- So one intersection implies infinite intersections.

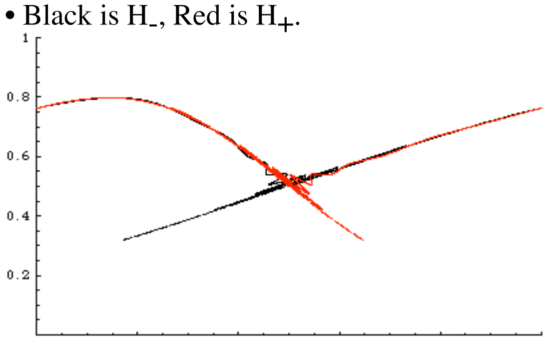


Tangles

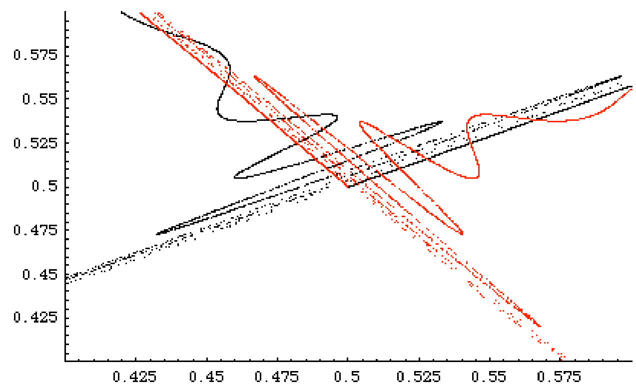
- Since H_+ and H_- cannot cross themselves, but must cross each other an infinite number of times, they must wind tightly about each other. In fact, the windings must be infinitely tight (i.e. between any two crossings there are an infinite number of other crossings).
- This results in a complex structure called a **tangle**. If the two manifolds are from the same hyperbolic point, this structure is called a **homoclinic tangle**. If they are from different hyperbolic points it is called a **heteroclinic tangle**. The intersections of the manifolds are called **homoclinic or heterclinic points.**

Homoclinic Tangle for K=0.8

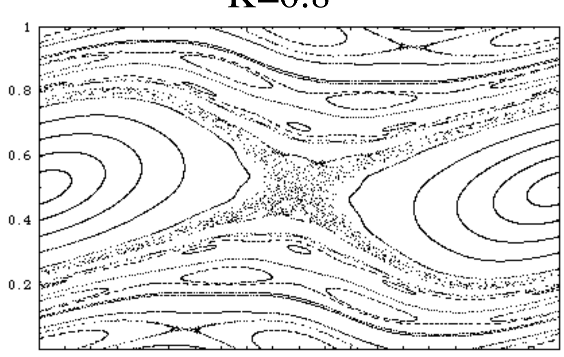
• For fixed point at (0.5,0), but the plot has been shifted up by 0.5 units.
• Black is H_-, Red is H_+.



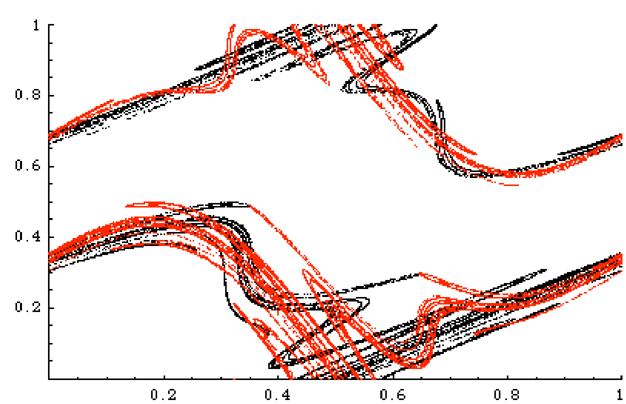
Detail of Last Slide



(Shifted) Surface of Section for K=0.8



Unshifted Tangle for K=1.5



Surface of Section for K=1.5