The Breakup of KAM Tori

The importance of being irrational

KAM Tori vs. Periodic Trajectories

- For an integrable system, we have seen that there are two types of motion: periodic and quasiperiodic.
- KAM tori are quasiperiodic trajectories with irrational winding numbers.
- Periodic trajectories are where phase-locking that can result in the formation of a nonlinear resonance can occur. These trajectories have rational winding numbers.

Winding Numbers for Standard Map

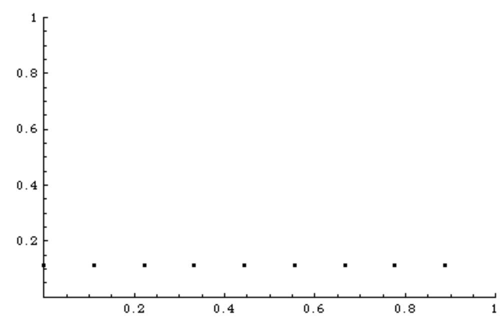
- For K=0, the standard map reduces to:

  $r_{n+1} = r_n$, mod 1

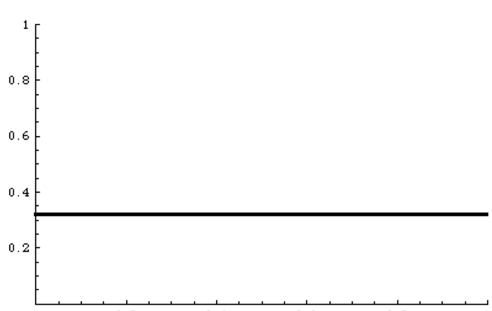
  $\theta_{n+1} = \theta_n + r_n$, mod 1
- From this we see that

  $\theta_n = \theta_0 + nr_0$, mod 1
- So every iteration of the map increases θ by r_0.
- This means the winding number for the Standard Map is just r_0.

K = 0, r_0 = 1/9



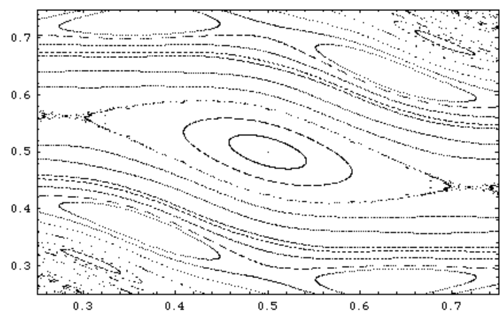
K = 0, r_0 = 1/π



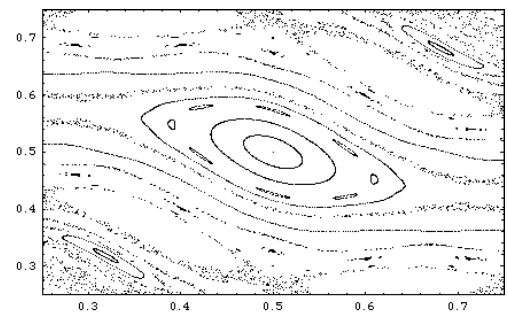
Breakup of KAM Tori

- As the perturbation is increased, two things happen.
  - The natural frequencies of the system are altered, which changes the winding number associated with a particular initial condition.
  - Nonlinear resonances form around trajectories with rational winding number.
- As the perturbation gets larger the resonances get bigger and start to break up the nearby KAM tori.

Detail for K = 0.8



Detail for K=0.95



How Irrational is an Irrational Number?

- KAM tori that are close to resonances will be the first to break up. Those that are far from resonances will be last to break up.
- This means that KAM tori with winding numbers that are "close to" a rational number will break up first. The last to break up will be those with winding numbers that aren't "close to" any rational number.
- So the KAM tori with the "most irrational" winding numbers are the last to break up.

Continued Fractions

- To determine which irrationals are closest to (or farthest from) a rational number we look at the irrational's continued fraction expansion.
- By truncating the continued fraction expansion we obtain rational approximates for the irrational number.
- Irrationals whose continued fraction expansions converge slowly are the "most irrational" (farthest from any nearby rational) and will be last to break up.

The Golden Torus

- The irrational number whose continued fraction expansion converges the slowest is the Golden Mean $(1+\sqrt{5})/2$.
- So the KAM torus whose winding number is the Golden Mean (the Golden Torus) is the last to break up.