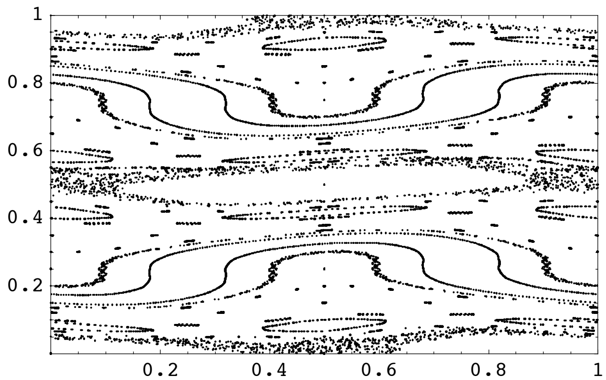
PHY 402 - Classical Mechanics II
Computational Project #3
Wes Taylor

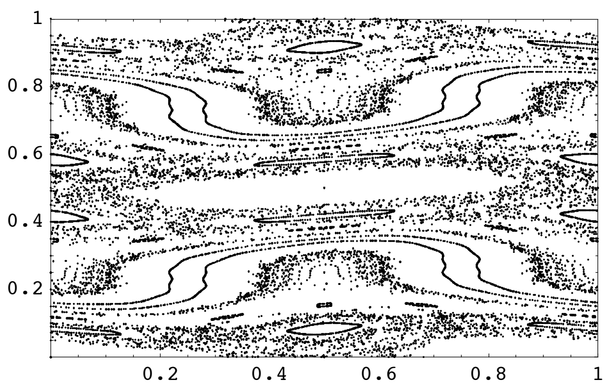
For this, our final computational project I will take you on a tour of classical mechanics using as our guide, the Berry Map. (insert stadium-like applause)
The Berry Map is defined as:
$\theta_{n+1} = \theta_n - \sin(2\pi r_{n+1})$, Mod 1
$r_{n+1} = r_n + \frac{k}{2\pi} \sin(2\pi\theta_n)$, Mod 1
where k is the perturbation parameter

To get an idea of what this looks like, we will create a few section surfaces using different perturbation parameters.

```
f[x_] := N[{Mod[x[[1]] - Sin[2 * Pi * (Mod[x[[2]] + (k / (2 * Pi)) * Sin[2 * Pi * x[[1]]], 1])], 1],
   Mod[x[[2]] + (k / (2 * Pi)) * Sin[2 * Pi * x[[1]]], 1]}]
k = 0.2;
MapData = Flatten[Table[
   Join[NestList[f, {0, 0.05 * i}, 400], NestList[f, {0.5, 0.05 * i}, 400]], {i, 0, 20}], 1];
ListPlot[MapData, PlotStyle → PointSize[0.004], PlotRange → {{0, 1}, {0, 1}},
 Frame → True, Axes → False]
```

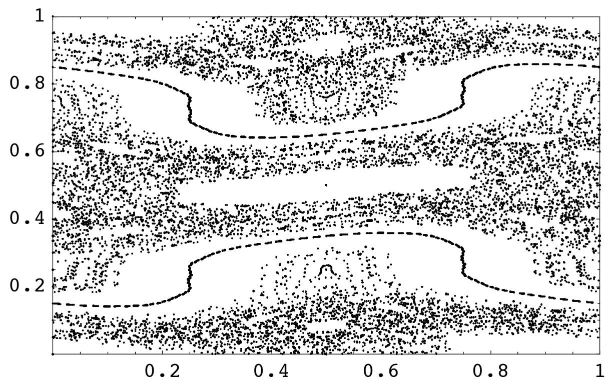


This is the map for k = 0.2. Now, here's k = 0.25



Between the first two sections you can easily see the most distinctive features of the map including the stable period one trajectory at (0.5,0.5), resonance state centered around (0,0) as well as other resonances throughout, four distinctive KAM tori, early signs of chaotic trajectories throughout, and, most interestingly, humps that alternate, centered on the stable period-one trajectories at (0.5,0.25), (0.5,0.75), (0,0.25) and (0,0.75). These humps are about the only "physical feature" that we have not witnessed with the Standard Map. We will come back to these humps later.

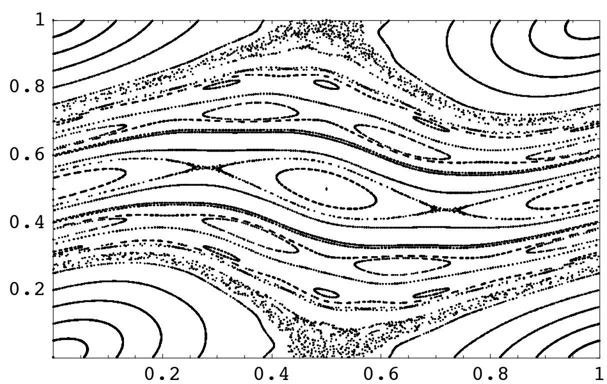
Back to the different perturbations, here is k = 0.3.



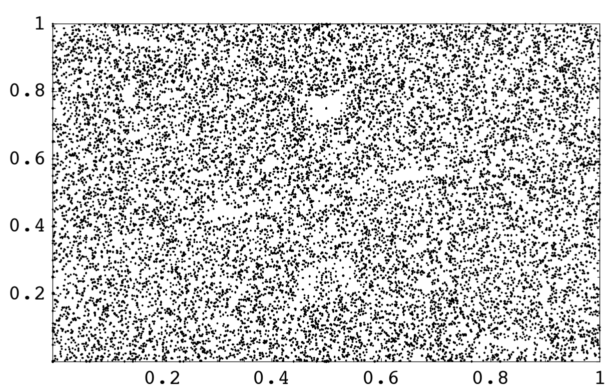
Notice how quickly the different KAM tori broke down into chaos. It appears here that there is only one KAM torus left, but, if you were to open the file WTPerturbationMovie on the desktop and animate the graphics, you would see that this is in fact two different tori which seem to come together and separate just before they break down. For your information, the movie begins with k=0, and runs through k=1.25 in increments of 0.01. From analyzing the map, it appears as if these last KAM tori are destroyed around k = 0.37. Since they are the last to be destroyed, we can assume that these are our Golden Tori for which their winding number is the most irrational number, the Golden Mean.

Note: I say that they are the Golden Tori, instead of the Golden Torus, because it appears that these two tori that kind of weave towards and away from each other break at the same time, leading me to believe that there is more than one torus with the Golden Mean for its winding number.

For a quick comparison, here is a plot of the Standard Map where k = 0.8,



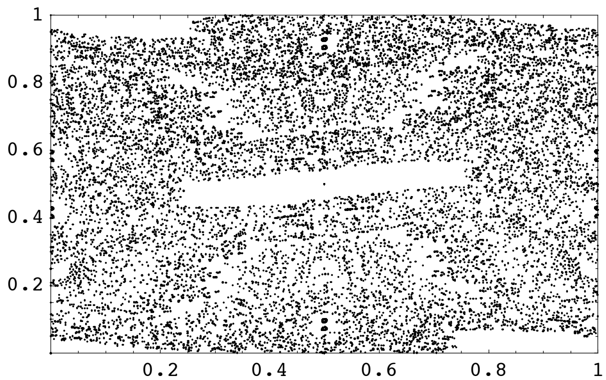
and here is the Berry Map for k = 0.8.



From this we see that not only does the Berry Map have those strange humps, but it also goes towards hard chaos much quicker than does the Standard Map.

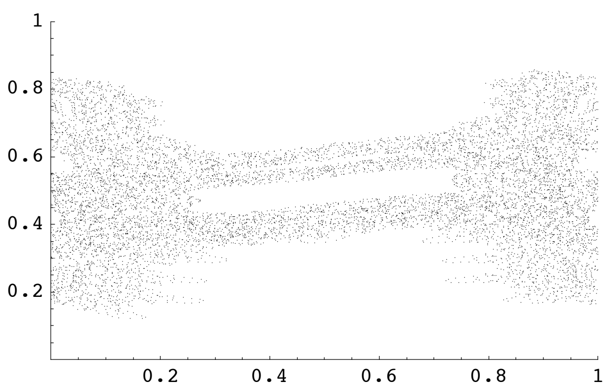
Since we know the different resonances and KAM tori are destroyed rather quickly, let's look at which values of k cause global and hard chaos.

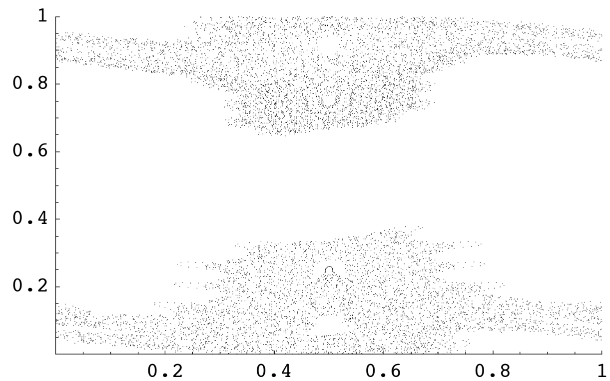
At k = 0.38, the last KAM tori have been destroyed and we have global chaos.



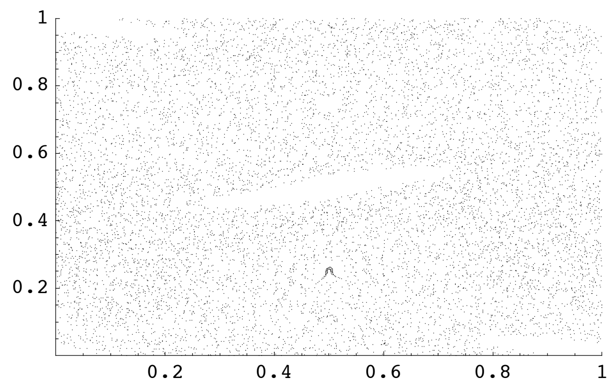
The interesting thing is, however, that at this point, certain trajectories are not allowed to cross the area where the KAM once stood. For example, here are the plots of two single trajectories, one located closer to the middle, between where the tori where, and the other, outside the tori location.

```
k = .38;
MapData = NestList[f, {0.2, 0.2}, 10000];
ListPlot[MapData, PlotStyle → PointSize[0.0004], PlotRange → {{0, 1}, {0, 1}}]
```

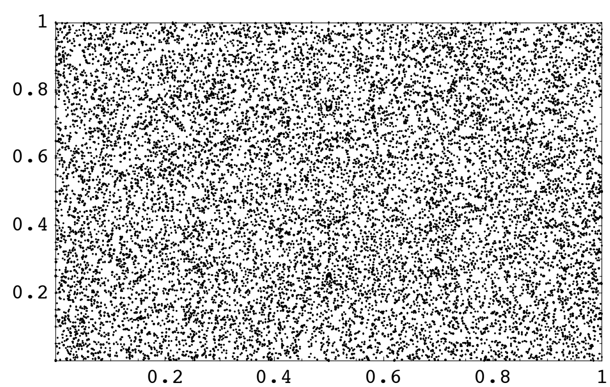




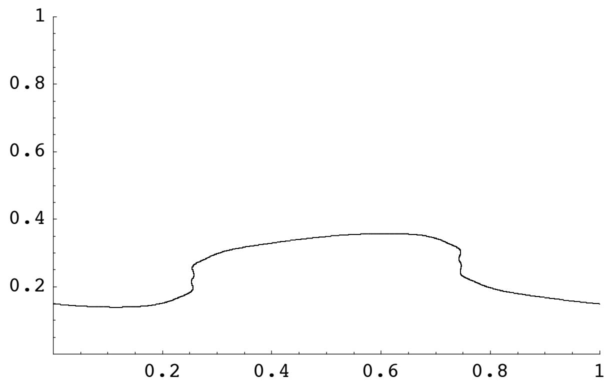
It is not until around k = 0.55 that a single trajectory is able to cross the phase space from top to bottom and left to right without any blockages.



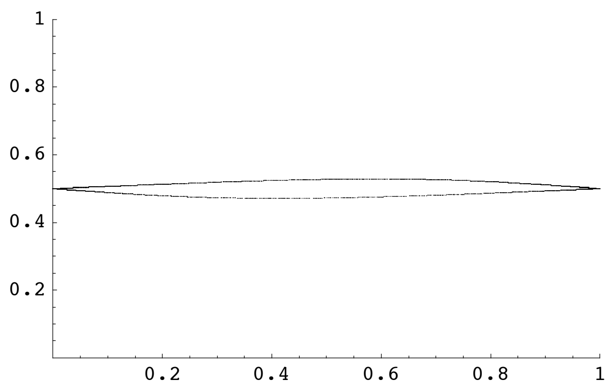
Now, for hard chaos. This one is much harder to determine from the plots. If you go back and watch the movie again you can see what appear to be two resonance states around the fixed points in the middle of the humps that seem to "beat" rather than be destroyed as the perturbation is increased. Despite this apparent illusion, it would seem that hard chaos begins around k = 0.92.



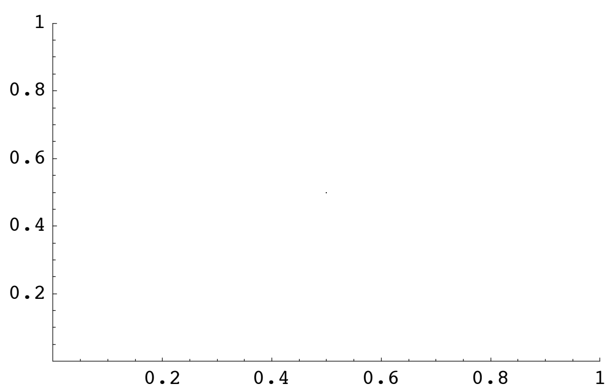
With the global chaos discussion above, we looked at two different chaotic trajectories. Let's now see if we can find some KAM, resonance, or periodic trajectories.
Here is a KAM trajectory created with the intial starting point of (0.1,0.14) at k = 0.3.



This is the one of the two main resonance trajectories created at k = 0.05 starting at (0,0.5)



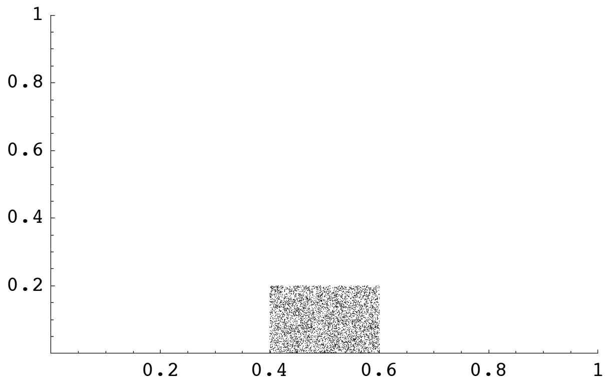
And, of course, a boring look at the trajectory of a fixed point



Wohoo!

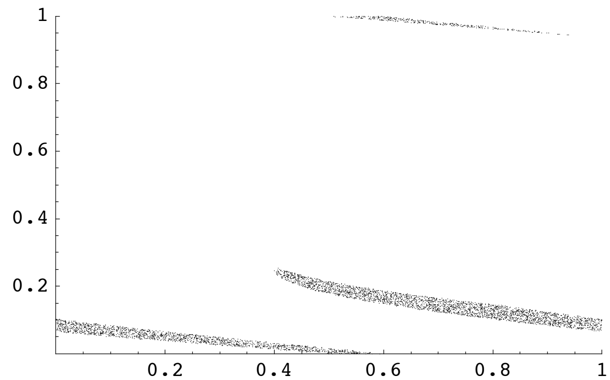
One major characteristic of Poincare Sections is that they preserve area. Let's first just look graphically to see if our map will follow this characteristic. We'll start with 5000 points located in a box that ranges in θ from 0.4 - 0.6, and r from 0 - 0.2.

```
MapData = Table[{Random[Real, {0.4, 0.6}], Random[Real, {0, 0.2}]}, {i, 0, 5000}];
ListPlot[MapData, PlotStyle → PointSize[0.0004], PlotRange → {{0, 1}, {0, 1}}]
```

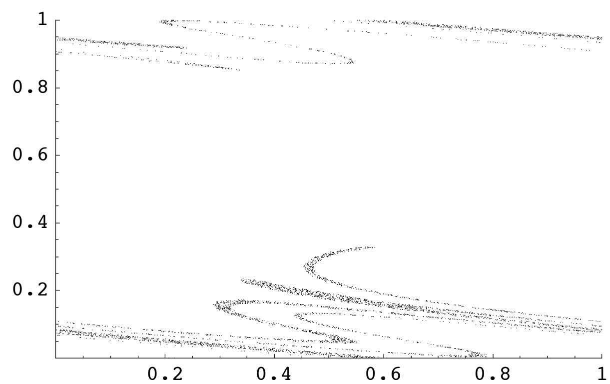


Now, here's how it looks after 1 iteration.

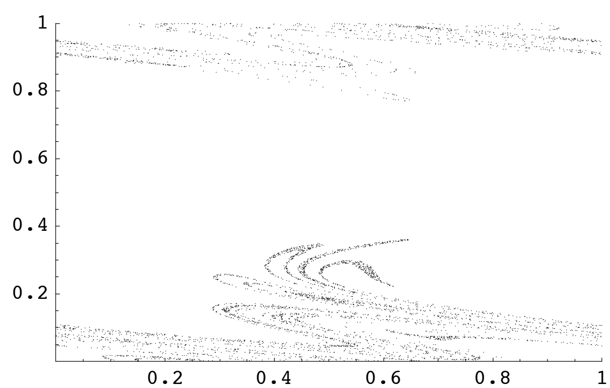
```
k = 0.6; MapData =
 Table[Nest[f, {Random[Real, {0.4, 0.6}], Random[Real, {0, 0.2}]}, 1], {i, 0, 5000}];
ListPlot[MapData, PlotStyle → PointSize[0.0004], PlotRange → {{0, 1}, {0, 1}}]
```



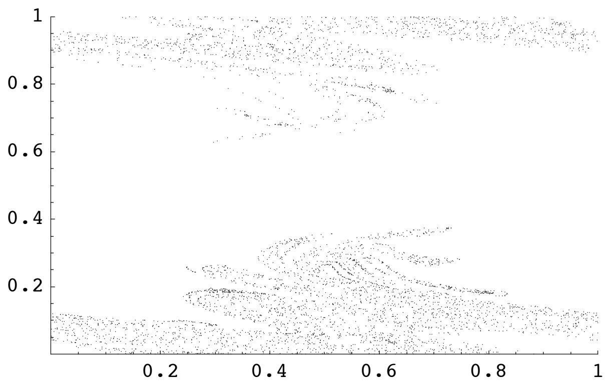
It definitely looks as if it could be preserving area. Let's watch for a few more iterations.
After 3



5...



and finally, 10 iterations.



It is hard to tell exactly from the plots if our map is indeed area preserving, so we will have to go to the math. For our map to be area preserving, the determinant of the Jacobian must equal 1. So let's have a look.

The following is my work for determining if the Berry Map is area preserving.

```
ClearAll[k, x, y, A, B, F, G, P, r]
```

```
r[x_, y_] := y + (k / (2 * Pi)) * Sin[2 * Pi * x];
θ[x_, y_] := x - Sin[2 * Pi * r[x, y]]
```

Here, r[x,y] stands for r_{n+1} , y for r_n , θ[x,y] for θ_{n+1} , and y for θ_n .
The work below is piecing together the different parts of the Jacobian were
G = $\frac{\partial r_{n+1}}{\partial r_n}$, F = $\frac{\partial r_{n+1}}{\partial \theta_n}$, B = $\frac{\partial \theta_{n+1}}{\partial r_n}$, A = $\frac{\partial \theta_{n+1}}{\partial \theta_n}$

```
A = D[(x - Sin[2 * Pi * r[x, y]]), x]
```

$$1 - 2\,k\,\pi\,\mathrm{Cos}[2\,\pi\,x]\;\mathrm{Cos}\left[2\,\pi\left(y + \frac{k\,\mathrm{Sin}[2\,\pi\,x]}{2\,\pi}\right)\right]$$

```
B = D[(x - Sin[2 * Pi * r[x, y]]), y]
```

$$-2\,\pi\,\mathrm{Cos}\left[2\,\pi\left(y + \frac{k\,\mathrm{Sin}[2\,\pi\,x]}{2\,\pi}\right)\right]$$

```
F = D[(y + (k / (2 * Pi)) * Sin[2 * Pi * x]), x]
```

$$k\,\mathrm{Cos}[2\,\pi\,x]$$

```
G = D[(y + (k / (2 * Pi)) * Sin[2 * Pi * x]), y]
```

$$1$$

There for our Jacobian will be
$$J = \begin{pmatrix} G & F \\ B & A \end{pmatrix}$$

```
J = {{G, F}, {B, A}}
```

$$\left\{\{1,\ k\,\mathrm{Cos}[2\,\pi\,x]\},\ \left\{-2\,\pi\,\mathrm{Cos}\left[2\,\pi\left(y + \frac{k\,\mathrm{Sin}[2\,\pi\,x]}{2\,\pi}\right)\right],\ 1 - 2\,k\,\pi\,\mathrm{Cos}[2\,\pi\,x]\;\mathrm{Cos}\left[2\,\pi\left(y + \frac{k\,\mathrm{Sin}[2\,\pi\,x]}{2\,\pi}\right)\right]\right\}\right\}$$

And, drumroll please, the determinant is...

```
Det[J]
```

```
1
```

There you have it folks, the determinant for the Jacobian of our map is 1, therefore it is area preserving.

Another important aspect of our map are the fixed points. A fixed point is defined as a point on that, when iterated through the map, gives itself as the next point. To determine which points to test to see if they are fixed points, we look at our mapping functions as well as the map itself. Doing this we find that the following appear to be fixed points:

```
(θ, r)
(0, 0)
(0.5, 0.5)
(0, 0.5)
(0.5, 0)
(0.5, 0.25)
(0.5, 0.75)
(0, 0.25)
(0, 0.75)
```

To determine if they are in fact fixed points, we run them through the map, where f is the mapping functions.

```
f[{0, 0}]
```

```
{0., 0.}
```

```
f[{0.5, 0.5}]
```

```
{0.5, 0.5}
```

```
f[{0, 0.5}]
```

```
{1., 0.5}
```

Note: Entering 0 for θ here gives us 1 after an iteration, but we know that on our map, 1 = 0. Isn't that great math?

```
f[{0.5, 1}]
```

```
{0.5, 0.}
```

Note: Likewise here, we get 0 instead of 1.

```
f[{0.5, 0.25}]
```

```
{0.5, 0.25}
```

```
f[{0.5, 0.75}]
```

```
{0.5, 0.75}
```

```
f[{0, 0.25}]
```

```
{0., 0.25}
```

```
f[{0, 0.75}]
```

```
{0., 0.75}
```

Indeed, all eight of the points we found by looking at the map and mapping functions were fixed points. We must now determine for which parameter values those fixed points are stable (elliptical) points, or unstable (hyperbolic) points. To do so, we take the trace of the stability matrix, P, at the fixed point for the different parameter values. If -2 < TrP < 2, then the fixed point is stable, else, it is unstable. Note: P is just equal to J evaluated at the fixed point.

For (0,0)

```
x = 0; y = 0; k = .1
Tr[J]
```

```
1.37168
```

As we see from above, at k = 0.1, (0,0) is stable. Now, let's see for what range of values each fixed point is stable or unstable.

	Stable	Unstable	Neither (TrP = 2 or -2)
(0,0)	0 < k < 0.63662	k > 0.63662	k = 0, k = 0.63662
(0.5,0.5)	0 < k < 0.63662	k > 0.63662	k = 0, k = 0.63662
(0,0.5)	0 < k < 0.0000005	k > 0.0000005	k = 0, k = 0.0000005
(0.5,0)	0 < k < 0.0000005	k > 0.0000005	k = 0, k = 0.0000005

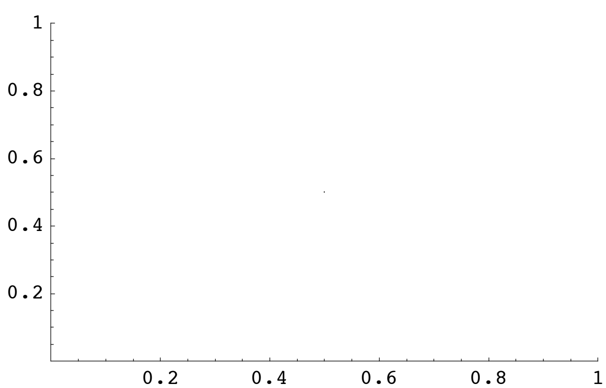
As you can see, the points at (0,0.5), and (0.5,0) are stable for only the slightest time in our movie. For the final four fixed points, something a little more interesting seems to occur.

(0.5,0.25)	none	none	all k
(0.5,0.75)	none	none	all k
(0,0.25)	none	none	all k
(0,0.75)	none	none	all k

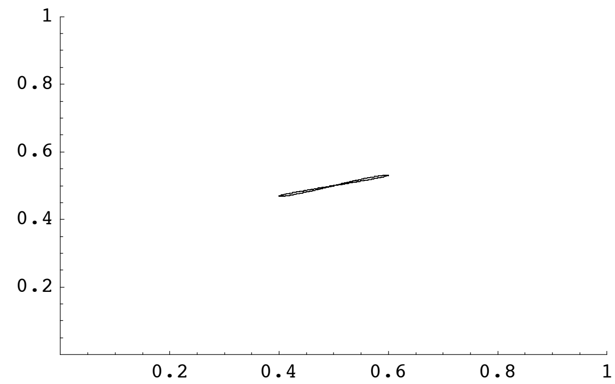
These fixed points seem to be neither stable nor unstable, rather they are of the variety that seem to be stable if they are moved to one side, and unstable if moved to the other.

Before we move on to our final look at the Berry Map, I would like to look at the fixed point located at (0.5,0.5). Going back to the perturbation movie, if you watch the center carefully, you see a sideways figure eight appear from the center then disappear into the chaos. That is what becomes of the (0.5,0.5) trajectory above k = 0.63662. Let's look at the building trajectory.

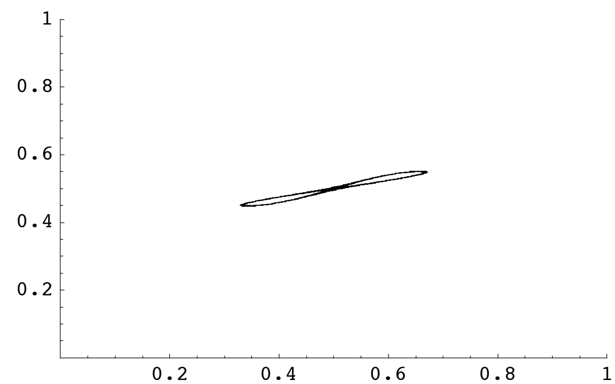
k= 0.63



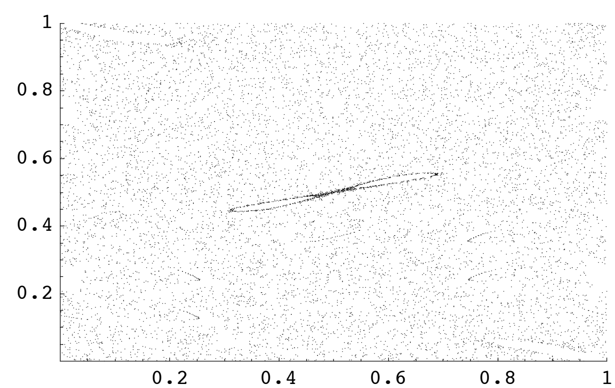
Now k = 0.66



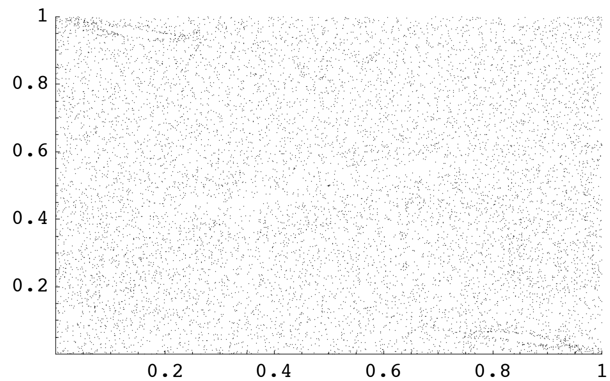
k = 0.7



k = 0.72



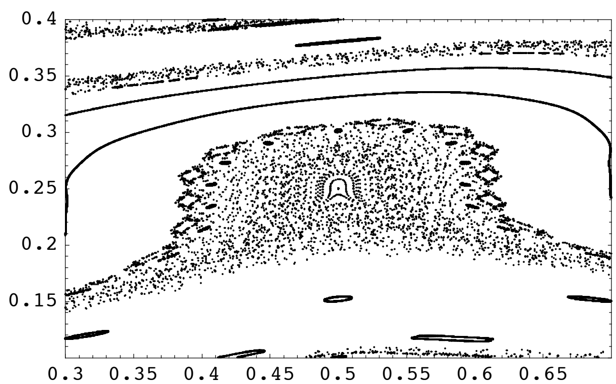
And finally, k = 0.77



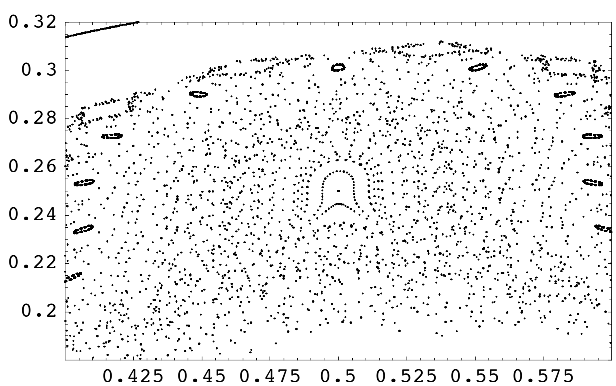
As you can see, that fixed point's trajectory is now chaotic.

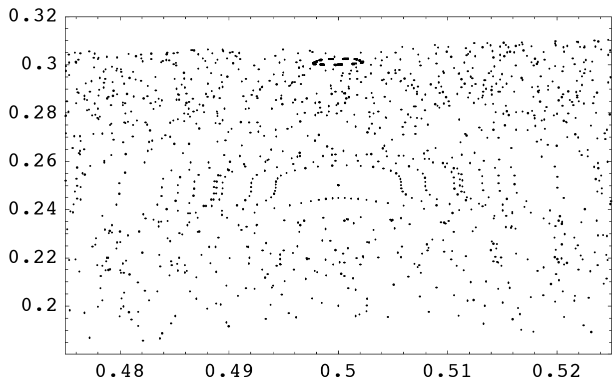
To wrap up this project, I wanted to take a closer look at one of the things that makes the Berry Map special--those wierd humps.

As you can see from the zoomed in plot below, the hump look more or less like a chaotic region with small resonances below and a good number floating on the edge of the chaos.



If we look closer,however,we will see a pattern.





The hump repeats on itself as it we zoom in closer. This is the fractal structure of the Berry Map, our own unique study.

That will just about do it for this project. If you would like to browse through the perturbation maps with parameter markings, feel free to look in the notebook, Comp3Perturbations located on the desktop. For your knowledge, I spent a good hour trying to show the stable and unstable manifolds, but was unsuccessful. Oh well.

I'm going on my 8th hour of working on this, my final comp project, and quite frankly, i'm a little sleepy. Hope you've enjoyed it.