

## Exploring The Driven Damped Pendulum

Derived from Section 6.3 of *Classical Mechanics with Maxima* by Todd Timberlake and J. Wilson Mixon, Jr. (Springer, 2015), [sites.berry.edu/ttimberlake/teaching/cm\\_maxima/](http://sites.berry.edu/ttimberlake/teaching/cm_maxima/).

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1. If you have not already done so, download and install *wxMaxima* from [wxmaxima.sourceforge.net](http://wxmaxima.sourceforge.net).
2. The equations of motion for our driven, damped pendulum system are:

$$\begin{aligned}\dot{x} &= v, \\ \dot{v} &= -2\beta v - \omega_0^2 \sin(x) + \gamma\omega_0^2 \cos(\omega t).\end{aligned}$$

We will explore the motion of this system for  $\omega = 2\pi$  rad/s,  $\omega_0 = 1.5\omega$ ,  $\beta = \omega_0/4$ , and  $\gamma = 1$  with initial values  $x_0 = -\pi/2$  and  $v_0 = 0$ . We can solve the system of ODEs using *Maxima*'s `rk` command, which implements a fourth-order Runge-Kutta method with fixed time step. The code to set the parameter values and solve the ODEs is:

```
omega:2*pi$ omega0:1.5*omega$ beta:omega0/4$ gamma:1$
data: rk( [v, -2*beta*v - omega0^2*omega0*sin(x) + gamma*omega0^2*omega0*cos(omega*t)],
  [x,v], [-%pi/2,0], [t, 0, 200, 0.01])$
```

The first line in the code above sets the parameter values. The second and third lines implement the `rk` command. The first argument of the `rk` command is a list that specifies the time derivatives of the dependent variables (as determined by the equations of motion). The second argument is a list of the dependent variables ( $x$  and  $v$ ). The third argument is a list of initial values for the variables ( $-\pi/2$  and 0). The final argument gives the independent variable, its initial value, its final value, and the step size. Here we are using a step size of 0.01 s and solving from  $t = 0$  to  $t = 200$  s.

Type this code into *Maxima* and evaluate it. Make sure you are not receiving any error messages. You can test the success of the computation by typing and evaluating “`data[20001]`”, which should show the output from `rk` for  $t = 200$  (the 20,001st time value). Note that the output is a list of three numbers: time  $t$ , angle  $x$ , and angular velocity  $v$ .

3. Next we want to use our solution to construct a plot of  $x$  versus  $t$ . We must first construct a list of  $(t, x)$  data points using *Maxima*'s `makelist` command. Then we can plot that set of points using the `draw2d` command. The code to accomplish these tasks is given below. (Note: the `wx` version of the `draw2d` command displays the plot within a *wxMaxima* notebook. Otherwise the plot will display in a separate window.)

```
txL: makelist( [data[i][1],data[i][2]] , i, 1, 1001)$
wxdraw2d( xaxis = true, xlabel = "t (s)", ylabel = "x (rad)",
  point_size = 0, points_joined = true, points(txL) )$
```

The first line of code above constructs our list of  $(t, x)$  ordered pairs. Note that `data[i]` is the  $i^{\text{th}}$  element of the output from `rk`, so each value of  $i$  corresponds to a different output time. The code `data[i][1]` is the first element (the  $t$  value) of the  $i^{\text{th}}$  output, while `data[i][2]` is the second element (the  $x$  value). The list `txL` only includes the first 1001 output points, so it only extends to  $t = 10$  s (1001 points with a step size of 0.01). The second and third lines of code create a plot of the list of  $(t, x)$  ordered pairs, with the points connected by a line.

Construct a plot of  $x$  versus  $t$  using your solution from `rk`. Check with your instructor to make sure your plot is correct. You should notice that the plot settles into a regular oscillation.

4. Now construct a plot of  $v$  versus  $t$  using your `rk` solution. Note that the  $v$  values are stored in `data[i][3]`. You should give your list of  $(t, v)$  ordered pairs a different name (maybe `tvL`) to keep it distinct from your list of  $(t, x)$  ordered pairs. You should also change the label on the  $y$ -axis of the plot. Check with your instructor to make sure your plot is correct. You should notice that the plot settles into a regular (but decidedly non-sinusoidal) oscillation.

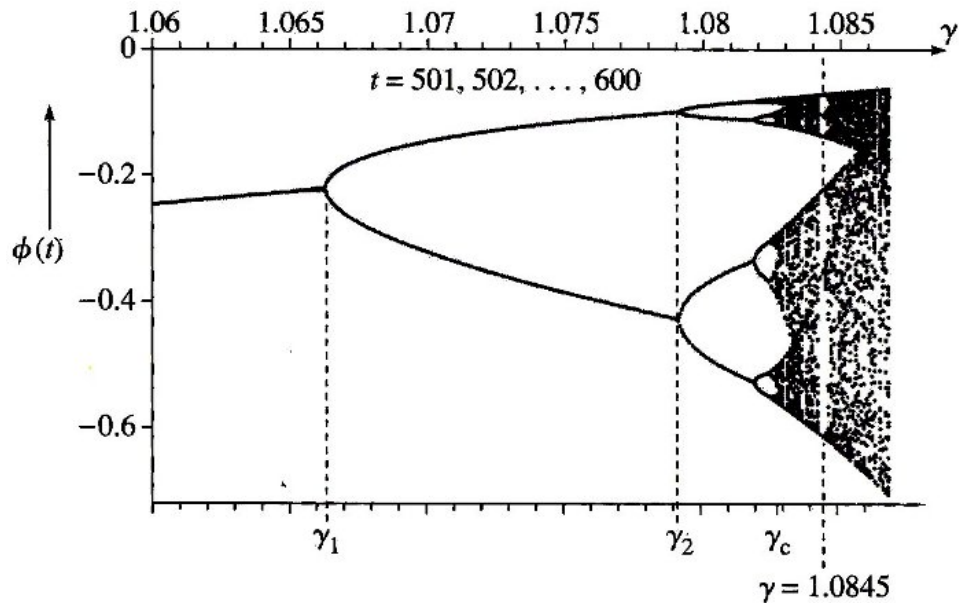
5. Construct a plot of the trajectory for this pendulum in phase space (angular velocity  $v$  versus angle  $x$ ). In this case we need to construct a list of ordered pairs  $(x, v)$  and then plot that list. Again, give your new list of ordered pairs a new name and use correct labels for your plot. This time extend your plot to  $t = 200$  s (so you should use the first 20,001 ordered pairs). Check with your instructor to make sure your plot is correct.
6. You should have found that the phase space trajectory shows some transient behavior before settling into what looks like a distorted ellipse. To focus on the steady-state motion and ignore the transient behavior we can redo the plot, this time starting our list of ordered pairs with the 5001<sup>st</sup> output value. The resulting plot should show the phase space trajectory from  $t = 50$  s to  $t = 200$  s.
7. The steady state motion of our pendulum seems to be periodic, but with what period? By analogy with the driven, damped harmonic oscillator we might guess that the period is the period of the driving torque. To check our guess we can sample the location of the pendulum in phase space after each cycle of the driving torque. A plot of these points is called a “strobe plot” because it only shows the system at discrete time intervals, much like viewing a moving object illuminated by a strobe light. To construct our strobe plot we need to show the phase space position at one second intervals (since the period of our driving torque is 1 s). To do this we simply add an extra argument to the `makelist` command to construct a list that uses every 100<sup>th</sup> point (since our time step is 0.01 s, 100 times steps is 1 s). Then we can plot the points using `draw2d`, but this time we don’t want to connect the dots. The code below shows you how to accomplish these tasks.

```
strobe:makelist( [data[i][2],data[i][3]], i, 5001, 20001, 100)$
wxdraw2d(xaxis = true, xlabel = "x (rad)", ylabel = "v (rad/s)",
  point_type = 7, point_size = 1, points(strobe1),
  xrange = [-%pi,%pi], yrange = [-20,20] )$
```

Use the code to construct your strobe plot. You should see only a single point. Really this is 150 points that all lie at the same location. This illustrates that the motion of this pendulum is indeed periodic with a period equal to 1 s (the period of the driving torque).

8. Now you have all of the code in place to explore the motion of the driven, damped pendulum for a variety of parameter values. Investigate the motion for  $\gamma = 1.075$  (all other values the same). How is the motion different from that with  $\gamma = 1$ ? In particular, how has the period of the motion changed?
9. Do the same for  $\gamma = 1.081$ . You may want to adjust the `xrange` and `yrange` of your strobe plot to get a better view of what is going on. How did the period of the motion change from  $\gamma = 1.075$  to  $\gamma = 1.081$ ?
10. Repeat your computations, this time using  $\gamma = 1.1$ . Is the motion periodic for this value of  $\gamma$ ? (Note that the points in the strobe plot are limited to a small region of phase space, but they do not seem to form a countable set. This structure is known as a *strange attractor*.)
11. Repeat your computations, this time using  $\gamma = 1.13$ . Is the motion periodic for this value of  $\gamma$ , or is it chaotic? If it is periodic, what is the period?

12. The plot shown below is known as a *bifurcation diagram*.<sup>1</sup> To construct a bifurcation diagram for our pendulum we plot the  $x$  (angle) values from our strobe plots, ignoring the transient motion at small times, for many different values of  $\gamma$ . In the plot,  $\phi$  represents the angle that we are calling  $x$ . Note that for some values of  $\gamma$  there seems to be a single curve. For those values the motion is periodic with the same period as the driving torque. Likewise, when there are two curves the motion is periodic with a period twice that of the driving torque (the strobe plot alternates between two different points in phase space). Describe how the bifurcation diagram shown below corresponds to the motion you saw for  $\gamma = 1, 1.075, 1.081, 1.1$  and  $1.13$ .



13. Next investigate the motion for  $\gamma = 1.5$ . What is different about the behavior of the  $x$  coordinate in this case? Explain what is happening with the motion of the pendulum.
14. To construct a strobe plot for the case  $\gamma = 1.5$  we must first compute the *principal value* of the  $x$  values. The principal value is just the angle, equivalent to  $x$ , that lies in the range from  $-\pi$  to  $\pi$ . To find the principal angle we use *Maxima*'s `mod` command. The command `mod(a,b)` gives the value of  $a$ , modulo  $b$ . To find the principal value of  $x$ , we must first add  $\pi$ , then find the value modulo  $2\pi$ . The result will be an angle in the range  $0$  to  $2\pi$ , but it is the wrong angle because we added  $\pi$  at the start. So we must then subtract  $\pi$  to get the correct principal value. The code below shows how to construct the list of ordered pairs for our strobe plot.

```
strobe: makelist( [mod( data[i][2]+%pi,2*%pi)-%pi, data[i][3]], i, 5001, 20001, 100)$
```

Construct this list and make a plot of the points. Is the motion periodic, or does the strobe plot display a strange attractor?

<sup>1</sup>This bifurcation diagram is taken from *Classical Mechanics* by John R. Taylor (University Science Books, 2005), p. 484).

15. Strange attractors are characteristic of *chaotic* motion in dissipative systems like the driven, damped pendulum. To qualify as chaotic, a system must display *sensitive dependence on initial conditions*. Technically, sensitive dependence means that trajectories that start with only slightly different initial conditions will diverge exponentially over time.

We can investigate this sensitive dependence in our pendulum system by looking at the motion of two pendula with  $\gamma = 1.5$ . We will keep all other parameters as before, but we will start one pendulum with  $x_0 = -\pi/2$  and the other with  $x_0 = -\pi/2 + 0.001$  (both will have  $v_0 = 0$ ). This tiny difference in initial angle should grow rapidly over time in this chaotic system and eventually the trajectories will diverge noticeably. Use the computational tools discussed above to construct a single plot showing  $x$  versus  $t$  for both of these pendula, for  $t = 0$  to 30 s. Note that you can include more than one `points` command within a single `draw2d` command.

16. You should have found that the trajectories of the two pendula diverge noticeably after about 10 seconds. Now go back and decrease the initial separation of the two pendula by a factor of ten. Do the trajectories still diverge noticeably? After how much time? Repeat this procedure until you convince yourself that each time you reduce the initial separation by a factor of ten you just delay the divergence of the trajectories by a few seconds.
17. The behavior you have just observed (decreasing separation by a *factor* of ten merely *adds* a few seconds to the divergence time) indicates that these trajectories are diverging exponentially. To prove this we can construct a plot of  $\ln |d|$  versus  $t$ , where  $d$  is the difference between the angles of the two pendula. Construct this plot (note that *Maxima* uses `log` for the natural logarithm function and `abs` for the absolute value function). You should find that the plot shows large fluctuations, but generally trends upward in a linear fashion. The slope of this general trend is known as the *Lyapunov exponent* ( $\lambda$ ) for the motion. Roughly speaking, the distance  $d(t)$  between the pendula at time  $t$  is given by  $d(t) \approx d(0)\exp(\lambda t)$ . As long as  $\lambda > 0$  then the trajectories are diverging exponentially and the motion is chaotic. If  $\lambda < 0$  then the trajectories are converging exponentially.
18. To find out more about the driven, damped pendulum and other classical systems that you can explore with the *Maxima* computer algebra system, check out *Classical Mechanics with Maxima* by Todd Timberlake and J. Wilson Mixon (Springer, 2016). You can find out more about this book at [sites.berry.edu/ttimberlake/teaching/cm\\_maxima/](http://sites.berry.edu/ttimberlake/teaching/cm_maxima/).