Hydrogen Atom Tutorial

We have seen that the radial equation for the hydrogen-like atom with Z protons can be written

$$\frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{\ell(\ell+1)}{\rho^2}\right] u$$

where

$$\rho \equiv \kappa r, \qquad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}, \qquad \text{and} \qquad \rho_0 \equiv \frac{Zme^2}{2\pi\epsilon_0\hbar^2\kappa}.$$

1. Write an approximate form for this differential equation in the limit $\rho \to \infty$.

2. Solve the approximate equation to find the asymptotic solution for $u(\rho)$ in the limit $\rho \to \infty$.

3. Apply the physicality conditions to find the physically relevant asymptotic solution.

4. Write an approximate form for the radial equation in the limit $\rho \to 0$.

5. Show that $u(\rho) = A\rho^{\ell+1} + B\rho^{-\ell}$ is the general solution for this approximate equation.

6. Apply the physicality conditions to find the physically relevant asymptotic solution.

7. Given our two asymptotic solutions we will define a new function $v(\rho)$ such that $u(\rho) \equiv \rho^{\ell+1} e^{-\rho} v(\rho)$. Substituting this into our original radial equation we get

$$\rho \frac{d^2 v}{d\rho^2} + 2(\ell + 1 - \rho) \frac{dv}{d\rho} + (\rho_0 - 2(\ell + 1))\, v = 0.$$

Now we will try to solve this differential equation using a power series method. Assume a solution of the form

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j.$$

Compute $dv/d\rho$. Rewrite the sum, if necessary, so that each term involves ρ^j and the sum starts at $j = 0$.

8. Compute $\rho(dv/d\rho)$. Rewrite the sum, if necessary, so that each term involves ρ^j and the sum starts at $j = 0$.

9. Compute $d^2 v/d\rho^2$.

10. Compute $\rho(d^2 v/d\rho^2)$. Rewrite the sum, if necessary, so that each term involves ρ^j and the sum starts at $j = 0$.

11. Substitute the sums you found above into the differential equation for $v(\rho)$.

12. Note that *each term* in the sum must be zero (do you understand why?). Use this fact to solve for c_{j+1} in terms of c_j.

13. In order to get a physically relevant (i.e. normalizable) wave function this series must terminate at some point. So there must be some value of j (let's call it j_{max}) such that for $j > j_{max}$ we have $c_j = 0$. Solve for this j_{max}. What is the minimum value of j_{max}?

14. We can define a new quantum number $n \equiv j_{max} + \ell + 1$. Solve for ρ_0 in terms of n. What is the minimum value of n?

15. Use the definition of ρ_0 to solve for κ in terms of n (and constants). To simplify this expression we can introduce a new constant called the *Bohr radius*

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx 0.529 \times 10^{-10} \text{ m}.$$

Write κ as a function of Z, a, and n.

16. Use the definition of κ to solve for E in terms of n (and constants). The solutions E_n are the energy eigenvalues of the hydrogen-like atom.