Dirac Notation Tutorial

1 Basis vectors and inner products

We begin by defining our *right basis vectors*:

$$|i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |j\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can represent a right vector in this basis as

$$|A\rangle = A_x|i\rangle + A_y|j\rangle,$$

where A_x and A_y are the (complex) coefficients of the expansion of the state $|A\rangle$ in the $i-j$ basis. We can then define the corresponding *left basis vectors* as

$$\langle i| = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad \langle j| = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

and then any left vector can be written as

$$\langle A| = A_x^*\langle i| + A_y^*\langle j|$$

where * indicates the complex conjugate of a number.

Use these definitions to evaluate the following *inner products*. Note that an inner product always combines a left vector and a right vector to produce a number. It is a generalization of the dot product with which you are already familiar.

1. $\langle i|i\rangle =$

2. $\langle j|j\rangle =$

3. $\langle i|j\rangle =$

4. $\langle i|A\rangle =$

5. $\langle A|i\rangle =$

6. $\langle A|A\rangle =$

7. If $|B\rangle = B_x|i\rangle + B_y|j\rangle$, then $\langle A|B\rangle =$

8. $\langle B|A\rangle =$

9. In general, how is $\langle A|B\rangle$ related to $\langle B|A\rangle$?

2 Operators and basis transformations

Now we can define some operators in this notation. Consider the operators $\hat{P}_x = |i\rangle\langle i|$ and $\hat{P}_y = |j\rangle\langle j|$.

1. Evaluate $\hat{P}_x|A\rangle$.

2. Evaluate $\hat{P}_y|A\rangle$.

3. Let $\hat{I} = \hat{P}_x + \hat{P}_y$. Show by evaluating $\hat{I}|A\rangle$ that $\hat{I}$ is the identity operator.

4. Suppose we want to write our vector $|A\rangle$ in a new basis, with right basis vectors $|n\rangle$ and $|m\rangle$. Let's start this process by expressing the identity operator in the new basis. Write this new version of the identity operator below.

5. Now let's apply this new version of the identity operator to the state $|A\rangle$, which was written in the old basis. Combine all of the $|n\rangle$ terms together and all of the $|m\rangle$ terms together.

6. Show that we get the same result if we multiply the vector

$$|A\rangle = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

by the matrix

$$\hat{O} = \begin{pmatrix} \langle n|i\rangle & \langle n|j\rangle \\ \langle m|i\rangle & \langle m|j\rangle \end{pmatrix}.$$

7. To get an idea of how this works, suppose we define our new basis such that $|n\rangle = i|j\rangle$ and $|m\rangle = -|i\rangle$. Rewrite $|A\rangle$ in this new basis.

8. Now we've introduced the idea of representing an operator with a matrix. Write the corresponding matrix for $\hat{P}_x$(using the $i - j$ basis) in the space below.

9. Write the corresponding matrix for $\hat{I}$ in the space below.

10. Now suppose we define an operator in the $i - j$ basis as

$$\hat{O} = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}.$$

Find the expectation value of this operator for the state $|A\rangle = 2|i\rangle - 3i|j\rangle$.

11. A matrix (or operator) is said to be *Hermitian* if you transpose the matrix, take the complex conjugate of the result, and get back the original matrix. Is the operator in the previous problem Hermitian?

12. Is the identity matrix Hermitian?

13. Is $\hat{P}_x$ Hermitian?

14. Is the basis transformation matrix (the matrix $\hat{O}$ in question 6 above) necessarily Hermitian?