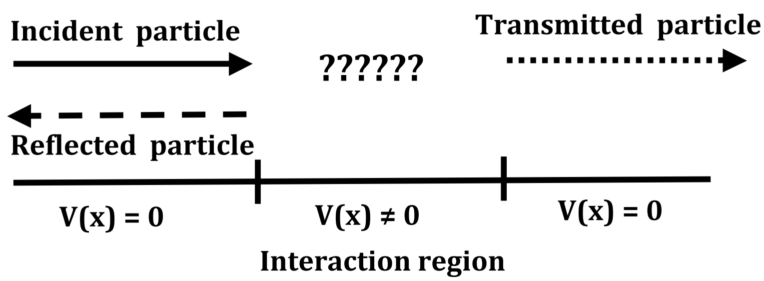
Scattering I: The Dirac Delta

1 The Basic Idea of Scattering in 1-D

- We imagine a situation in which a particle starts far to the left in a region ($x < x_l$) where there is no interaction (so $V(x) = 0$). The particle moves to the right and eventually enters a region ($x_l \leq x \leq x_r$) where there is an interaction ($V(x) \neq 0$). The end result of this interaction is that the particle can be reflected back to the left (so that it re-enters the region $x < x_l$ as a free particle) or it can be transmitted into a region on the right ($x > x_r$) in which there is no interaction ($V(x) = 0$).



- If the left region we represent the particle as a plane wave with components traveling to the right (representing the incident wave) and to the left (representing the reflected wave):

$$\psi(x) = Ae^{ikx} + Be^{-ikx},\ x < x_l$$

  where $k = \sqrt{2mE}/\hbar$ and $E > 0$.

- In the right region we represent the particle as a plane wave traveling to the right only (representing the transmitted wave):

$$\psi(x) = Ce^{ikx},\ x > x_r$$

- In the interaction region ($x_l \leq x \leq x_r$) $\psi(x)$ will be found by solving the EEP for $E > 0$ with potential $V(x)$.

- By matching the solutions in different regions with appropriate boundary conditions we can determine B and C in terms of A and k. From these we can find the probability for reflection ($|B|^2/|A|^2$) and the probability for transmission ($|C|^2/|A|^2$).

- Note that we use the same k for all three plane waves. Why?

- We know that we can't really represent particles as plane waves because these states aren't physical (they aren't normalizable). We really should use *wave packets* peaked at some value of k instead. However, this plane wave analysis is much simpler and is a good approximation to the more correct wave packet analysis in most cases.

2 Example I: The Delta Well

- For the potential $V(x) = -\alpha\delta(x)$ there is no interaction except at $x = 0$ (so $x_l = x_r = 0$). Write down the general scattering solution for $\psi(x)$ as a piecewise function in the two regions.

- Find an equation relating A, B, and C from the continuity of $\psi(x)$ at $x = 0$.

- Recall that since the potential is infinite as $x = 0$ we can have a discontinuity in $d\psi/dx$ at that point. This discontinuity must satisfy:

$$\lim_{\epsilon \to 0} \left(\frac{d\psi}{dx}\bigg|_{\epsilon} - \frac{d\psi}{dx}\bigg|_{-\epsilon} \right) = \frac{2m}{\hbar^2} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} V(x)\psi(x)dx.$$

  Use this condition to find C as a function of A and k.

- Now put your two equations together to determine B as a function of A and k.

- Calculate the transmission coefficient $T = |C|^2/|A|^2$. Express this as a function of energy E.

- Calculate the refelection coefficient $R = |B|^2/|A|^2$. Express this as a function of energy E.

- Show that $T + R = 1$.

- Evaluate T and R in the limit $E \to \infty$. Compare this with the classical result for this system ($T = 1$, $R = 0$).

- Evaluate T and R in the limit $E \to 0$. Does this result make sense? Explain.

3 Example II: The Delta Barrier

- Suppose we make our delta function potential repulsive (so that it is a barrier) rather than attractive. This just means $V(x) = \alpha\delta(x)$. Note that all we've done is to change α to $-\alpha$. Except for this change, all of our results above will still hold. Write down T and R for this delta barrier.

- Think about T and R in the limit $E \to \infty$ in this case. How does this compare with the classical result for the delta barrier ($T = 0$, $R = 1$)?

- Think about T and R in the limit $E \to 0$ in this case. Does this result make sense? Explain.