Scattering II: The Finite Square Well and Barrier

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1 Finite Square Well

Consider the scattering with $E > 0$ for the finite square well:

$$V(x) = \begin{cases} 0, & |x| > a \\ -V_0, & |x| \leq a. \end{cases}$$

We have already determined the solution for the EEP in the region inside the well ($\psi(x) = C\sin(\ell x) + D\cos(\ell x)$, where $\ell = \sqrt{2m(E+V_0)}/\hbar$). In the regions to the left and right of the well we use our normal scattering solutions. So the resulting solution is

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < -a \\ C\sin(\ell x) + D\cos(\ell x), & -a \leq x \leq a \\ Fe^{ikx}, & x > a \end{cases}$$

where $k = \sqrt{2mE}/\hbar$.

1. Write down the equation for continuity of ψ at $x = -a$.

2. Write down the equation for continuity of $d\psi/dx$ at $x = -a$.

3. Solve for A as a function of C and D. This should not take very much work.

4. Solve for B as a function of C and D. This should take even less work after you've done the previous question.

5. Write down the equation for continuity of ψ at $x = a$.

6. Write down the equation for continuity of $d\psi/dx$ at $x = a$.

7. Now we could solve for C and D in terms of F and then substitute these values back into the equations we previously obtained for A and B. Then we could solve for B in terms of A and F in terms of A. I won't make you do the algebra. Here's the result:

$$\frac{B}{A} = \frac{(\ell^2 - k^2)\sin(2\ell a)}{2k\ell\cos(2\ell a) - i(\ell^2 + k^2)\sin(2\ell a)}$$

and

$$\frac{F}{A} = \frac{2k\ell}{2k\ell\cos(2\ell a) - i(\ell^2 + k^2)\sin(2\ell a)}.$$

Use these results to calculate R and T.

8. Show that $R + T = 1$.

9. Evaluate the limits of R and T as $E \to \infty$. Does this make sense? Note that classically we have $R = 0$ and $T = 1$.

10. Evaluate the limits of R and T as $E \to 0$. Does this make sense?

11. It turns out that something unusual happens when the particle has an energy that corresponds to an energy eigenvalue of the *infinite* well. These energies are

$$E_n = -V_0 + \frac{n^2\pi^2\hbar^2}{8ma^2}$$

(which looks slightly different from our earlier result for the ISW, but only because here the well has width $2a$ instead of a and the bottom of the well is at $-V_0$ instead of 0). Calculate the value of ℓ at these energies.

12. What are the values of R and T if the incident particle has one of these values of ℓ? This phenomenon is known as a *scattering resonance*.

2 The Finite Square Barrier

Consider a square barrier potential given by

$$V(x) = \begin{cases} 0, & |x| > a \\ V_0, & |x| \leq a. \end{cases}$$

If $E > V_0$ then our results are the same as for the square well, but with $V_0 \to -V_0$ (which basically just changes the value of ℓ). For the case $0 < E < V_0$ our scattering wave function is

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < -a \\ Ce^{\alpha x} + De^{-\alpha x}, & -a \leq x \leq a \\ Fe^{ikx}, & x > a \end{cases}$$

where $k = \sqrt{2mE}/\hbar$ and $\alpha = \sqrt{2m(V_0 - E)}/\hbar$. If we follow the same boundary matching procedure that we used for the square well above we find:

$$T = \frac{(2k\alpha)^2}{(k^2 + \alpha^2)^2 \sinh^2(2\alpha a) + (2k\alpha)^2}$$

and

$$R = \frac{(k^2 + \alpha^2)^2 \sinh^2(2\alpha a)}{(k^2 + \alpha^2)^2 \sinh^2(2\alpha a) + (2k\alpha)^2}.$$

(Note that $\sinh(x) = (e^x - e^{-x})/2$).

1. Evaluate T and R in the limit of a tall ($\alpha \to \infty$) and wide ($a \to \infty$) barrier. Does this result make sense?

2. What do you think will happen in the case where the barrier is not too tall (so V_0 is not that much greater than E) and not too wide (so a and αa are no too large)?