Identical Particles Tutorial

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1. Consider a wave function representing a system of two identical particles. Suppose we solve the EEP for this system and write the eigenstates as $\psi(\mathbf{r}_1, \mathbf{r}_2)$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ represent the positions of the first and second particles. Now consider the "particle exchange operator" P. This operator simply exchanges the locations of the two particles. Since the two particles are identical, this shouldn't have any real effect on the system.[1] What do you get if you apply this operator to the state $\psi(\mathbf{r}_1, \mathbf{r}_2)$?

2. What do you get if you apply P^2 to $\psi(\mathbf{r}_1, \mathbf{r}_2)$?

3. We know that $[P, H] = 0$ because the Hamiltonian can't possibly change under the exchange of two identical particles.[2] Therefore it should be possible to find simultaneous eigenstates of H and P. Let's assume that $\psi(\mathbf{r}_1, \mathbf{r}_2)$ is such a simultaneous eigenstate. What are the possible eigenvalues for P for this state?

4. So we find that simultaneous eigenstates of P and H must satisfy $\psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\psi(\mathbf{r}_2, \mathbf{r}_1)$. It turns out that the $\pm$ has physical significance. Particles with half-integer spin (called *fermions*) have the minus sign, while particle with integer spin (called *bosons*) have the plus sign. Now suppose we have two identical bosons, each of which is in a well-defined state. The two states are designated ψ_a and ψ_b. Show that the simple product of $\psi_a(\mathbf{r}_1)$ and $\psi_b(\mathbf{r}_2)$ is *not* generally an eigenstate of the particle exchange operator.

5. What about $\psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)$? Is it an eigenstate of the particle exchange operator?

6. Construct a linear combination of these two product states that *is* an eigenstate of the particle exchange operator with the proper eigenvalue (remember these particles are bosons). Show that the state you have constructed is an eigenstate with the correct eigenvalue.

[1] Note that in quantum mechanics, when we say two particles are identical we mean that they are truly indistinguishable. There is no way to tell one electron from another electron. All electrons are fundamentally identical.

[2] Usually the Hamiltonian depends on $|\mathbf{r}_1 - \mathbf{r}_2|$, in which case it is obviously invariant under the exchange of the two particles.

7. Assuming ψ_a and ψ_b are orthonormal, normalize the eigenstate of P that you found in the previous question.

8. What happens if the two particle are in exactly the same state (say, ψ_a)? Write the resulting wave function below. [Note: you may need to rethink the normalization.]

9. Now what if these particles are fermions? Construct a linear combination of these two product states that is an eigenstate of the particle exchange operator with the proper eigenvalue (remember these particles are bosons). Show that the state you have constructed is an eigenstate with the correct eigenvalue. Normalize the wave function.

10. What happens if the two particle are in exactly the same state (say, ψ_a)? Write the resulting wave function below.

11. Is it possible for two identical fermions to be in the same quantum state? [Note: this is the famous *Pauli Exclusion Principle*. It follows directly from the symmetry of the two-particle wave function under the exchange of particles.]

12. Now consider a two particle harmonic oscillator system in which the particles don't interact with each other at all. The single-particle energy eigenstates are $\psi_n(x)$ and eigenvalues are $E_n = (n + 1/2)\hbar\omega$. If the two particles are bosons, what is the lowest energy eigenstate for this system and what is the energy eigenvalue of that state? [Note: introduce the variables x_1 and x_2 for the locations of the two particles, but consider the symmetrization requirements above.]

13. What is the first excited state and what is its eigenvalue? Is there more than one state with this energy?

14. Now consider the case where the particles are fermions. What is the lowest energy state and what is its eigenvalue?

15. What is the first excited state for the two fermions, and what is the eigenvalue for this state? Is there more than one state with this energy?