Harmonic Oscillator: Algebraic Solution

Let's summarize our results from last time:

- We found the fundamental commutation relation: $[x, p] = (xp - px) = i\hbar$.
- We defined ladder operators: $a_{\pm} = (2\hbar m\omega)^{-1/2}(\mp ip + m\omega x)$.
- We found the ladder operator commutator: $[a_-, a_+] = (a_-a_+ - a_+a_-) = 1$.
- We rewrote the HO Hamiltonian: $H = \hbar\omega(a_-a_+ - 1/2) = \hbar\omega(a_+a_- + 1/2)$.

Now suppose we have a state ψ which we know is an eigenstate of H with eigenvalue E: $H\psi = E\psi$. In the space below, show that the state $a_-\psi$ is also an eigenstate of H and determine its corresponding eigenvalue. (Hint: try to evaluate $H(a_-\psi)$ by rewriting H in terms of the ladder operators and using the ladder commutator to shuffle things around until you get H next to ψ.)

You should have found that the state $a_-\psi$ is an eigenstate with energy less than E. So by applying a_- to an energy eigenstate we are taken to an eigenstate with lower energy. This is why this is the "lowering" operator. But we can't keep this up forever because otherwise we would get into negative energies. So there must be a lowest energy eigenstate, which we will call ψ_0. If we try to apply a_- to this state it should give us a non-physical result. In this case it gives us $a_-\psi_0 = 0$. Use the definition of the lowering operator, $a_- = (2\hbar m\omega)^{-1}(ip + m\omega x)$, and the momentum operator, $p = -i\hbar(\partial/\partial x)$, to rewrite $a_-\psi_0 = 0$ as a differential equation. Then solve the differential equation.

Determine the energy eigenvalue for ψ_0. (Hint: just rewrite H in terms of the ladder operators and then hit ψ with it to see what you get.

To find the other eigenstates we just apply the raising operator a_+. We found earlier that if ψ is an eigenstate with eigenvalue E, then $a_-\psi$ is an eigenstate with eigenvalue $E - \hbar\omega$. Similarly, we could show that $a_+\psi$ is an eigenstate with eigenvalue $E+\hbar\omega$, so a_+ takes us *up* the ladder of eigenstates in the same way a_- took us down. It turns out that if ψ_0 is normalized then the (properly normalized) n^{th} energy eigenstate is given by

$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0.$$

To use this in practice we just use our function for ψ_0 that we found above (properly normalized) and express a_+ in terms of x and $\partial/\partial x$, then apply a_+ to ψ_0 n times. What is the energy eigenvalue for the state ψ_n?

With some more work we can find that $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$ and $a_-\psi_n = \sqrt{n}\psi_{n-1}$. Show that this implies $a_+a_-\psi_n = n\psi_n$ and $a_-a_+\psi_n = (n+1)\psi_n$.

We can use these relations, along with the fact that the energy eigenstates are orthonormal ($\int \psi_n^*\psi_m dx = \delta_{nm}$), to find expectation values. For example, we can express x and p in terms of ladder operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-), \qquad\qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-).$$

Use these to evaluate $\langle x\rangle$ and $\langle x^2\rangle$.

Extra Problems (Just for Fun!)
Show that $a_+\psi$ is an eigenstate of H with eigenvalue $E + \hbar\omega$.

Given $\psi_n = (1/\sqrt{n!})(a_+)^n\psi_0$, prove that $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$.

Prove that $a_-\psi_n = \sqrt{n}\psi_{n-1}$.