Spherical Harmonics Tutorial

The solutions to the angular equation are given by

$$Y_\ell^m(\theta,\phi) = \epsilon\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}e^{im\phi}P_\ell^m(\cos\theta)$$

where

$$\epsilon = \begin{cases} (-1)^m, & m > 0 \\ 1, & m \le 0 \end{cases}$$

and

$$P_\ell^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|}P_\ell(x)$$

with

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell.$$

1. Find $P_1(x)$
2. Find $P_1^1(x)$.
3. Find $P_1^1(\cos\theta)$ and simplify if possible.
4. Find $Y_1^1(\theta,\phi)$.
5. Find $P_2(x)$.
6. Find $P_2^0(x)$.

7. Find $P_2^0(\cos\theta)$ and simplify if possible.

8. Find $Y_2^0(\theta,\phi)$.

9. Show that these two spherical harmonics are orthogonal. In other words, show that

$$\int_0^{2\pi}\int_0^{\pi}\left(Y_1^{-1}(\theta,\phi)\right)^* Y_2^0(\theta,\phi)\sin\theta d\theta d\phi = 0.$$