Momentum and Uncertainty

In this tutorial you will be introduced to the momentum operator, calculate expectation values and standard deviations for momentum, and see the "Uncertainty Principle" in action.

1. In quantum mechanics, all physically observable quantities are represented by operators that act on the wave function of a system to produce a new function. For example, the position observable x is represented by the operator $\hat{x}$ which acts on the wave function $\Psi(x,t)$ to produce the new function $x\Psi(x,t)$. (Note that I did not call this new function a wave function - that's because it likely doesn't satisfy the Schrödinger equation, and therefore wouldn't qualify as a wave function.) The position operator, and any operator composed solely out of the position operator, is pretty easy. But what about momentum? We would like to define the momentum operator in such a way that the expectation values satisfy the classical relation: $\langle \hat{p} \rangle = m d\langle x \rangle / dt$. Use the definition of the expectation value to show that this is equivalent to

$$\langle \hat{p} \rangle = m \int_{-\infty}^{\infty} x \left(\Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t} \Psi \right) dx.$$

2. Use the Schrödinger equation to show that

$$\langle \hat{p} \rangle = \frac{i\hbar}{2} \int_{-\infty}^{\infty} x \frac{\partial}{\partial x} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) dx.$$

3. Use integration by parts, and the fact that $\lim_{x\to\pm\infty} \Psi = 0$ (so that Ψ is square integrable and can be normalized) to show that

$$\langle \hat{p} \rangle = -\frac{i\hbar}{2} \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x} dx + \frac{i\hbar}{2} \int_{-\infty}^{\infty} \frac{\partial \Psi^*}{\partial x} \Psi dx.$$

4. Integrate the second term above by parts and use what you know about the wave function at infinity to show that

$$\langle \hat{p} \rangle = -i\hbar \int_{-\infty}^{\infty} \Psi^* \frac{\partial \Psi}{\partial x} dx.$$

5. Show that this result is consistent with defining the *momentum operator* as

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}.$$

6. Find $\langle \hat{p} \rangle$ for the wave function in the last tutorial:

$$\Psi(x,t) = \begin{cases} e^{-iEt/\hbar}\sqrt{\frac{15}{512a}}\left(4-(x/a)^2\right), & |x| \leq 2a \\ 0, & |x| > 2a. \end{cases}$$

7. Find $\langle \hat{p}^2 \rangle$ for the same wave function.

8. Calculate $\sigma_p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$.

9. Find the product $\sigma_p \sigma_x$ for this wave function (using your result for σ_x from the last tutorial). Show that it is consistent with Heisenberg's "Uncertainty Principle": $\sigma_p \sigma_x \geq \hbar/2$.