Harmonic Oscillator: Series Solution

In this tutorial you will solve the EEP for the harmonic oscillator with $V(x) = -kx^2$. The solution will employ the "method of Frobenius," or the power series method. This is a powerful method for solving differential equations which we will see again later in this course.

1. The EEP for the harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 = E\psi$$

where $\omega = \sqrt{k/m}$. We can change variables to $\xi = x\sqrt{m\omega/\hbar}$ and then the EEP equation becomes

$$\frac{d^2\psi}{d\xi^2} = \left(\xi^2 - \frac{2E}{\hbar\omega}\right)\psi.$$

First we will consider the asymptotic limit of this equation as $\xi \to \pm\infty$. Write the approximate equation that is valid for this limit.

2. Show that

$$\psi(\xi) = Ae^{-\xi^2/2} + Be^{\xi^2/2}$$

is the solution to the approximate equation.

3. Use physicality conditions to determine which of the constants (A and B) *must* be zero, and explain why it must be zero.

4. Now we will assume that the full EEP has a solution of the form

$$\psi(\xi) = h(\xi)e^{-\xi^2/2}$$

so that it has the appropriate form in the asymptotic limit. But we still need to find $h(\xi)$ that satisfies the EEP. Substituting this expression for ψ into the full EEP and canceling common factors we find

$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + \left(\frac{2E}{\hbar\omega} - 1\right)h = 0.$$

We will attempt to solve this equation by assuming that h can be represented as a power series in ξ:

$$h(\xi) = \sum_{j=0}^{\infty} a_j \xi^j.$$

First we will need to evaluate derivatives. Find the series that represents $dh/d\xi$.

5. Find the series that represents $\xi(dh/d\xi)$.

6. Find the power series that represents $d^2h/d\xi^2$.

7. Think carefully about the first couple of terms in the series above. Should they all really be there? Are any of them zero? If so, rewrite the initial value of the index so that none of the terms is zero.

8. Rewrite your series for $d^2h/d\xi^2$ in terms of the index $i = j - 2$.

9. Note that i (like j before) is a "dummy index", meaning it is just a placeholder that takes on different values in the sum. There is no reason we can't replace i with a different symbol. Rewrite your series from the previous question, but replace i with j.

10. Note that you now have series expressions for $d^2h/d\xi^2$, $\xi(dh/d\xi)$, and h. All of these expressions are sums from $j = 0$ to $j = \infty$, with each term proportional to ξ^j. Go back to the differential equation for h and substitute these series expression in the proper places. Combine like terms to get an expression of the form $\sum_{j=0}^{\infty}[\ldots]\xi^j = 0$.

11. Explain why the coefficient of ξ^j must vanish for *each* value of j.

12. Set the coefficient equal to zero and solve for a_{j+2}. This result is known as a *recursion relation.* It can give you the values of a_j, for all even j, in terms of a_0. Likewise, the recursion relation can give you the values of a_j, for all odd j, in terms of a_1.

13. Unfortunately there is a physical problem with the series solution generated by the recursion relation above. If either the odd or even coefficients are allowed to continue indefinitely then the wave function will not be normalizable. We can get rid of all even coefficients by setting $a_0 = 0$. Likewise, we can get rid of all odd coefficient by setting $a_1 = 0$. But we can't do both (or else $h = 0$ and $\psi = 0$, which is bad). Let's say we let $a_0 = 0$ and $a_1 \neq 0$. Then we need $a_{n+2} = 0$ for some odd n. Use the recursion relation to find the values of E (for any n) that will ensure that this condition is met.

The same equations, but for even values of n, ensures that we will have a normalizable solution with $a_1 = 0$ and $a_0 \neq 0$. So we have two sets of solutions, both with energy eigenvalue $E_n = \hbar\omega(n + 1/2)$:

Even solutions in which $a_j = 0$ for all odd j, and $a_j = 0$ for all even $j > n$ (for some even n). The non-zero coefficients can be written (using the recursion relation) in terms of a_0. The value of a_0 can be found by requiring that $\psi(\xi) = h(\xi)e^{-\xi^2/2}$ be properly normalized. These solutions will have even spatial symmetry about $x = 0$.

Odd solutions in which $a_j = 0$ for all even j, and $a_j = 0$ for all odd $j > n$ (for some odd n). The non-zero coefficients can be written (using the recursion relation) in terms of a_1. The value of a_1 can be found by requiring that $\psi(\xi) = h(\xi)e^{-\xi^2/2}$ be properly normalized. These solutions will have odd spatial symmetry about $x = 0$.