Hydrogen Atom II Tutorial

The solutions for $v(\rho)$ that satisfy the recursion relation we found last time can be written as

$$v(\rho) = L^{2\ell+1}_{n-\ell-1}(2\rho)$$

where

$$L^p_{q-p}(x) \equiv (-1)^p \left(\frac{d}{dx}\right)^p L_q(x)$$

and

$$L_q(x) \equiv e^x \left(\frac{d}{dx}\right)^q (e^{-x}x^q).$$

Note that $L^p_{q-p}(x)$ is called an *associated Laguerre polynomial* and $L_q(x)$ is a *Laguerre polynomial*.

1. $L_q(x)$ is a polynomial of degree ______________.

2. We can't allow $L^q_{q-p}(x) = 0$ because then our whole wave function would be zero (which is unphysical). What restriction does this place on the value of p, given q?

3. Use this restriction to determine the possible values for ℓ given n.

4. In the space below describe the possible values for all three quantum numbers: n, ℓ, and m. There is actually one more quantum number that has to do with spin: $m_s = \pm 1/2$. As you can see it only has two possible values. Still, there are lots of possible combinations for a given n. Since the energy depends only on n this means we have lots of *degeneracies* (different energy eigenstates that have the same energy eigenvalue).

Now we are ready to write out our full solution for the energy eigenstates of hydrogen-like atoms:

$$\psi_{n\ell m} = \sqrt{\left(\frac{2Z}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}} \left(\frac{2Zr}{na}\right)^\ell \exp\left(-\frac{Zr}{na}\right) \left[L^{2\ell+1}_{n-\ell-1}\left(\frac{2Zr}{na}\right)\right] Y^m_\ell(\theta,\phi).$$

These are, of course, orthonormal:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi^*_{n\ell m}(r,\theta,\phi)\psi_{n'\ell'm'}(r,\theta,\phi) r^2 \sin\theta \; dr \; d\theta \; d\phi = \delta_{nn'}\delta_{\ell\ell'}\delta_{mm'}.$$

5. Construct $\psi_{300}(r, \theta, \phi)$ for hydrogen. First, write down the appropriate spherical harmonic from Table 4.3 on p. 139 of the text.

6. Which associated Laguerre polynomial will you need? In other words, what are the values of $(n-\ell-1)$ and $(2\ell+1)$?

7. Now compute the Laguerre polynomial that you need.

8. Compute the associated Laguerre polynomial that you need.

9. Now write down the energy eigenstate wave function and simplify if possible.