Finite Square Well Tutorial

The finite square well is a system defined by the potential function

$$V(x) = \begin{cases} -V_0, & |x| < a \\ 0, & |x| \geq a. \end{cases}$$

In this tutorial we will investigate the energy eigenvalues and eigenstates for *bound states* ($E < 0$) in this system.

1. For the region with $|x| > a$ the EEP is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = \kappa^2\psi$$

where $\kappa \equiv \sqrt{-2mE}/\hbar$. This equation should now be sufficiently familiar that you can just write down the solution for the regions $x < -a$ and $x > a$. Write these below.

2. For $|x| < a$, show that the EEP can be written as

$$\frac{d^2\psi}{dx^2} = -\ell^2\psi$$

where $\ell \equiv \sqrt{2m(E + V_0)}/\hbar$.

3. Write down the general solution for this differential equation. (Hint: use trig functions.) Are there any physicality restrictions on this solution?

4. Because the potential is symmetric about $x = 0$ we know (because we proved it in a homework problem) that the energy eigenfunctions will be functions of even or odd symmetry. Consider the solutions you found for each of the three regions. Write a piecewise function (with three pieces) that is consistent with these solutions but also possesses even symmetry about $x = 0$.

5. Now do the same for odd symmetry.

6. Let's focus on the solutions with even symmetry. Write down the two equations that follow from making ψ continuous at $x = \pm a$. Are these two equations independent?

7. Now determine an equation that guarantees continuity of $d\psi/dx$ at $x = \pm a$.

8. Show that you can combine your answers to the last two questions to get the eigenvalue equation $\kappa = \ell \tan(\ell a)$.

9. Now we will define a new variable $z = \ell a$ and a new constant $z_0 = (a/\hbar)\sqrt{2mV_0}$. Show that $\kappa = (1/a)\sqrt{z_0^2 - z^2}$.

10. Find an equation for E as a function of z.

11. Show that the eigenvalue equation can be written

$$\tan(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}.$$

12. In the space below make a sketch of the function $\tan(z)$ for $0 < z < 3\pi$ and, separately, a sketch of the function $\sqrt{(z_0/z)^2 - 1}$ for $0 < z < z_0$.

13. Consider the limiting case of small z_0. Is there an even-symmetry bound state in this case? If so, what is the approximate energy eigenvalue in the limit $z_0 \to 0$?

14. Now let's consider the case of very large z_0. In this case the function $\sqrt{(z_0/z)^2 - 1}$ will be large for small values of z. So this function will intersect $\tan(z)$ near its vertical asymptotes. Use this fact to find an approximate formula for the energy eigenvalues E_n, valid as long as z is much smaller than z_0.

15. The energy eigenvalues for an infinite square well of width $2a$ are given by

$$E_n = \frac{n^2 \pi^2 \hbar^2}{8ma^2}.$$

How does this compare with your approximate formula for the energy eigenvalues in the finite square well when z_0 is large? Why might we expect these results to be similar?