Generalized Uncertainty Principle Tutorial

In Section 3.5 Griffith's proves the *Generalized Uncertainty Principle*:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle [\hat{A}, \hat{B}] \right\rangle \right)^2$$

where A and B are any two physical observables and σ_A and σ_B are the corresponding uncertainties of those observables. Recall that $\sigma_A \equiv \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$ and $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$. In this tutorial we will explore some applications of this generalized uncertainty principle to specific pairs of observables.

1. First let's make sure we can reproduce the usual $x - p$ uncertainty principle. Show that we can in the space below.

2. Any two observables are considered *complementary* (or incompatible) if it is impossible for BOTH observables to simultaneously have zero uncertainty. If A and B are compatible (not complementary) observables, then what must be the value of $[\hat{A}, \hat{B}]$? What can we say about the value of the commutator if A and B are complementary observables?

3. An eigenstate of A and also be an eigenstate of B if and only if ______________.
   (a) $[\hat{A}, \hat{B}] = 0$
   (b) $[\hat{A}, \hat{B}] = 1$
   (c) $[\hat{A}, \hat{B}] \neq 0$
   (d) $[\hat{A}, \hat{B}] = \infty$

4. Let's work through an example. Let's start by finding $[\hat{x}, \hat{H}]$ where $H = p^2/(2m) + V(x)$ is the Hamiltonian (total energy). To evaluate the commutator, use a test function $f(x)$ and the fact that $\hat{p} = -i\hbar d/dx$. Show your work below.

5. Now work out the position-energy uncertainty principle. Write it in the form $\sigma_x \sigma_H \geq$ ____.

6. Use the previous result to determine $\langle \hat{p} \rangle$ for *any* energy eigenstate in *any*system.

7. Let's take this in a slightly different direction and look at the time derivative of the expectation value of an operator. We want to examine

$$\frac{d}{dt}\langle \hat{Q} \rangle = \frac{d}{dt}\langle \Psi | \hat{Q} | \Psi \rangle.$$

Start by applying the chain rule to write this time derivative as the sum of three terms. (Note: it should make sense that you can do this if you think about writing the expectation value as an integral.)

8. Now use the the Schrödinger equation (which can be written as $i\hbar|d\Psi/dt\rangle = \hat{H}|\Psi\rangle$) and the fact that if $|A\Psi\rangle = \hat{A}|\Psi\rangle$ then $\langle A\Psi| = \langle\Psi|\hat{A}^\dagger$ to show that

$$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\left\langle[\hat{H},\hat{Q}]\right\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle.$$

9. If $\hat{Q}$ commutes with $\hat{H}$, what can we say about the expectation value of $\hat{Q}$?

10. Show that if we let $\hat{Q} = \hat{x}$ we get the same result for $\langle[\hat{x},\hat{H}]\rangle$ as before.

11. Now suppose we let $\Delta E = \sigma_H$ be the uncertainty in a particle's energy. We can't define the uncertainty of time because time is a parameter, not an operator, in quantum mechanics. But we can assume that we measure the passing of time by measuring the change of some other observable. In order to really tell that some time has passed we need the expectation value of some observable to change by an amount greater than the uncertainty of that observable. So we can define

$$\Delta t = \frac{\sigma_Q}{|d\langle Q\rangle/dt|}.$$

Show that if Q doesn't depend explicitly on time we get $\Delta E \Delta t \geq \hbar/2$.

12. The $n = 3$ state of hydrogen has a lifetime of about 10^{-8} s (so it takes about this long, on average, for the electron to drop down to a lower energy state and emit a photon). What is the uncertainty in the energy of the $n = 3$ state? How does this compare to the binding energy of the state $E_3 = 1.511$ eV?