

Adding Angular Momentum Tutorial

Recall our definition for the spin matrices:

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad S^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For this tutorial we will also introduce a new notation for the spin up and spin down states:

$$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

1. Evaluate the following results and give your answer in the new notation:

- (a) $S_z \uparrow =$
- (b) $S_z \downarrow =$
- (c) $S_x \uparrow =$
- (d) $S_x \downarrow =$
- (e) $S_y \uparrow =$
- (f) $S_y \downarrow =$
- (g) $S^2 \uparrow =$
- (h) $S^2 \downarrow =$

2. Now consider a situation in which we have two particles with spin-1/2 (like a hydrogen atom, in which both the proton and electron have spin-1/2). Suppose the electron is spin-up and the proton is spin-down. We can represent this state as $\uparrow\downarrow$. There are three other possible states. Write these states below using the new notation.

3. Now consider the operator $S_z = S_z^{(1)} + S_z^{(2)}$, where $S_z^{(1)}$ is an S_z operator that acts only on the first particle (the electron), while $S_z^{(2)}$ acts only on the second particle. Evaluate $S_z \uparrow\uparrow$. Is $\uparrow\uparrow$ an eigenstate of S_z ? If so, what is its eigenvalue?

4. Evaluate $S_z \uparrow\downarrow$. Is $\uparrow\downarrow$ an eigenstate of S_z ? (Think carefully about this.) If so, what is its eigenvalue.

5. Evaluate $S_z \downarrow\uparrow$. Is $\downarrow\uparrow$ an eigenstate of S_z ? If so, what is its eigenvalue.

6. Evaluate $S_z \downarrow\downarrow$. Is $\downarrow\downarrow$ an eigenstate of S_z ? If so, what is its eigenvalue.

7. So all four of these states are eigenstates of S_z , but two of them have the same eigenvalue. What's going on? To figure it out we need to look at the operator

$$\begin{aligned} S^2 &= (\mathbf{S}_1 + \mathbf{S}_2) \cdot (\mathbf{S}^{(1)} + \mathbf{S}^{(2)}) \\ &= (S^{(1)})^2 + (S^{(2)})^2 + 2\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} \\ &= (S^{(1)})^2 + (S^{(2)})^2 + 2S_x^{(1)}S_x^{(2)} + 2S_y^{(1)}S_y^{(2)} + 2S_z^{(1)}S_z^{(2)}. \end{aligned}$$

Evaluate $S^2 \uparrow\uparrow$. Is $\uparrow\uparrow$ an eigenstate of S^2 ? If so, what is its eigenvalue?

8. Evaluate $S^2 \uparrow\downarrow$. Is $\uparrow\downarrow$ an eigenstate of S^2 ? If so, what is its eigenvalue?

9. Evaluate $S^2 \downarrow\uparrow$. Is $\downarrow\uparrow$ an eigenstate of S^2 ? If so, what is its eigenvalue?

10. Evaluate $S^2 \downarrow\downarrow$. Is $\downarrow\downarrow$ an eigenstate of S^2 ? If so, what is its eigenvalue?

11. So $\uparrow\uparrow$ and $\downarrow\downarrow$ are simultaneous eigenstates of S_z and S^2 . But $\uparrow\downarrow$ and $\downarrow\uparrow$ are not eigenstates of S^2 (even though they are both eigenstates of S_z with eigenvalue 0). Let's consider linear combinations of these two states. Evaluate $S^2(\uparrow\downarrow + \downarrow\uparrow)$. Is this state an eigenstate of S^2 ? If so, what is its eigenvalue?

12. Evaluate $S^2(\uparrow\downarrow - \downarrow\uparrow)$. Is this state an eigenstate of S^2 ? (Think carefully about this.) If so, what is its eigenvalue?

13. You should have found three states whose eigenvalues for S^2 correspond to $s = 1$. Determine the value of the quantum number m for each state and then complete the table below by writing the (properly normalized) combination of single particle states in the correct spot.

$ sm\rangle$	written in terms of single-particle states
$ 1-1\rangle$	
$ 10\rangle$	
$ 11\rangle$	

14. You should have found one state whose eigenvalue for S^2 corresponds to $s = 0$. Determine the value of the quantum number m for this state and then complete the table below by writing the (properly normalized) combination of single particle states in the correct spot.

$ sm\rangle$	written in terms of single-particle states
$ 00\rangle$	

15. Write the state $\uparrow\downarrow$ in terms of the new $|sm\rangle$ states.
16. So we see that if we have a particle in a state $|s_1 m_1\rangle$ and another in a state $|s_2 m_2\rangle$, the combination of the two particles can form a variety of states $|sm\rangle$. We must have $m = m_1 + m_2$, but s can take on different values ranging from a minimum of $|s_1 - s_2|$ to the maximum value $s_1 + s_2$, in unit steps. For example, if we have a particle with $s_1 = 3/2$ and another particle with $s_2 = 1$ what are the possible values of s ?
17. Suppose we have a particle with $s_1 = 3/2$ and another particle with $s_2 = 1$. We want to know what combination of single-particle states will give us the eigenstate $|s = \frac{3}{2} \ m = -\frac{1}{2}\rangle$. To determine this we use the formula

$$|sm\rangle = \sum_{m_1+m_2=m} C_{m_1 m_2 m}^{s_1 s_2 s} |s_1 m_1\rangle |s_2 m_2\rangle$$

where the constants $C_{m_1 m_2 m}^{s_1 s_2 s}$ are called Clebsch-Gordan coefficients. You can find a table of these coefficients on page 188 of the textbook. Read the table caption and try to figure out the coefficients you need to write the $|\frac{3}{2} - \frac{1}{2}\rangle$ state in terms of the single-particle states. [Hint: you need the values from a single *column* of the table.]

18. The tables work the other way too. Suppose we want to write the state $|s_1 = \frac{3}{2} \ m_1 = -\frac{1}{2}\rangle |s_2 = 1 \ m_2 = 1\rangle$ as a linear combination of the $|sm\rangle$ states. We just use the formula

$$|s_1 m_1\rangle |s_2 m_2\rangle = \sum_{m_1+m_2=m} C_{m_1 m_2 m}^{s_1 s_2 s} |sm\rangle.$$

Try to determine the appropriate linear combination using Table 4.8. [Hint: you need the values from a single *row* of the table.]

19. Suppose you have a particle with spin-1/2 and another with spin-2. Their combined state is $|\frac{5}{2} \ \frac{1}{2}\rangle$. If you measure the $S_z^{(1)}$ (where particle 1 has spin-1/2), what values could you get? Which one is the most likely?

20. This procedure for adding spin angular momentum can be applied to other types of angular momentum. For example, if you have a spin-up electron in a hydrogen atom state with $\ell = 2$ and $m = -1$, what possible states of total angular momentum $|JM\rangle$ could the electron be in? What is the probability that it would be found in each of these states?

21. An electron with spin-up is in the hydrogen atom state ψ_{320} . What is the probability that this electron would be found in the total angular momentum state $|\frac{5}{2} - \frac{1}{2}\rangle$? [DO NOT USE the table of Clebsch-Gordan coefficients for this question! You don't need them.]