Free Particle Tutorial

Let's summarize our results for the free particle:

- Energy eigenstates: $\psi_k(x) = Ae^{ikx}$ where $k \equiv \pm\frac{\sqrt{2mE}}{\hbar}$.
  - These eigenstates cannot be normalized and therefore they are *unphysical*.
  - Solution to SE: $\Psi_k(x,t) = Ae^{ik(x-\hbar kt/(2m))}$ represents a plane wave with a *phase velocity* of $(\hbar k/(2m))$, which is half of the velocity of a classical particle with momentum $p = \hbar k$.
  - No free particle can exist in an energy eigenstate, but we can still use the energy eigenstates as a basis for constructing physically allowable states for the particle.
- Solution to general free particle SE: $\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\exp\left[i(kx - \frac{\hbar k^2}{2m}t)\right]dk$.
- Inverse Fourier transform to find $\phi(k)$ from initial wave function: $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}dx$.
  - As long as $\Psi(x,0)$ is properly normalized, $\Psi(x,t)$ will be properly normalized for all t.

Now let's see how this works in an example. Suppose a particle is in the ground state of an infinite square well with walls at $x = 0$ and $x = a$. If the walls suddenly disappear entirely, find $\Psi(x,t)$ for the particle.

1. What is the initial wave function $\Psi(x,0)$?

2. Write down the inverse Fourier transform integral that you need to evaluate to find $\phi(k)$.

3. Use Euler's formula ($e^{ix} = \cos x + i\sin x$) to rewrite the complex exponential and write $\phi(k)$ as the sum of two integrals (one of which evaluates to a real number and the other evaluates to an imaginary number).

4. Now evaluate this integral. You may find the following trig identities helpful:

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right], \qquad \sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)].$$

5. Now use your result for $\phi(k)$ to write the solution to the SE ($\Psi(x,t)$) in the form of an integral over k. You do not need to evaluate this integral (you can try to evaluate it on Mathematica if you like - but you won't get very far). Note: this may seem disappointing that we can get an explicit function for $\Psi(x,t)$. But in practice this kind of solution is still useful since this integral can be evaluated pretty easily (on a computer) for any given values of x and t. This type of solution, where the answer is left in the form of an integral, is known as *solution by quadrature*.

6. **Challenge:** Show that your solution for $\Psi(x,t)$ given above is properly normalized for all times. Here are some hints on how to do this:

   - Write $\Psi(x,t)$ as an integral over k. Write $\Psi^*(x,t)$ as an integral over k'.
   - Multiply these two together and integrate over all x (as you would normally do to check normalization).
   - Do the x integration *first*, pulling out any factors that don't depend on x.
   - Make use of the fact that
     $$\int_{-\infty}^{\infty} e^{i\alpha x} dx = 2\pi\delta(\alpha),$$
     where δ is the Dirac Delta function.
   - Make use of the fact that
     $$\int_{-\infty}^{\infty} f(z)\delta(z-\beta)dz = f(\beta).$$
   - You should then be able to reduce your result to a single integral of the form
     $$\int_{-\infty}^{\infty} \frac{1+\cos u}{(\pi^2-u^2)^2} du = \frac{1}{2\pi}.$$