Dirac Delta Function Tutorial

The Dirac Delta Function is not really a function. Technically it is a *distribution*, or generalized function. But it turns out to be quite useful in physics. Here is how it is defined:

$$\delta(x) = \lim_{a \to 0} \begin{cases} \frac{1}{2a}, & -a \le x \le a \\ 0, & |x| > a. \end{cases}$$

The Dirac delta function has the following special property:

$$f(x)\delta(x-c) = f(c)\delta(x-c)$$

for any x. It is also clear from the definition that

$$\int_a^b \delta(x)dx = \begin{cases} 1, & a \le 0 \le b \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\int_a^b f(x)\delta(x-c)dx = \begin{cases} f(c), & a \le c \le b \\ 0, & \text{otherwise.} \end{cases}$$

Let's practice with the delta function a bit. Evaluate the following integrals.

1. $\int_{-4}^{6}(x^2 - 3x + 2)\delta(x+3)dx$.

2. $\int_{-\infty}^{\infty} \exp(i\pi x)\delta(x-27)dx$.

3. $\int_0^{12} x^3 \exp(x^2 - 7x - 14)\delta(x-15)dx$.

Now we will consider a quantum system with a delta function potential:

$$V(x) = -\alpha\delta(x).$$

1. Note that for $x < 0$ the potential is zero and the EEP for bound states ($E < 0$) gives

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = \kappa^2\psi$$

   where $\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$. What is the solution for the bound state EEP in this region? Keep in mind that the eigenstates must be normalizable.

2. What is the solution for the bound state EEP in the region $x > 0$?

3. We require that $\psi(x)$ be continuous, which means $\lim_{x\to 0^+}\psi(x) = \lim_{x\to 0^-}\psi(x) = \psi(0)$. Use your solutions in the two regions to write down an equation for the boundary condition $\lim_{x\to 0^+}\psi(x) = \lim_{x\to 0^-}\psi(x)$.

4. Write down an expression for $\psi(0)$.

5. Now write out a piecewise solution for $\psi(x)$ in all three regions ($x < 0$, $x = 0$, and $x > 0$), making use of your boundary condition results.

6. Note that the potential changes from being finite to being infinite at $x = 0$, so we cannot assume that $d\psi/dx$ will be continuous here. But if it is discontinuous it must satisfy:

$$\frac{d\psi}{dx}\Big|_{+\epsilon} - \frac{d\psi}{dx}\Big|_{-\epsilon} = \frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} V(x)\psi(x)dx.$$

   Use your solution for $\psi(x)$ to determine the value of $\frac{d\psi}{dx}\big|_{+\epsilon}$.

7. Now use your solution to find $\frac{d\psi}{dx}\big|_{-\epsilon}$.

8. Now use the definition of the potential ($V(x) = -\alpha\delta(x)$) and your solution for $\psi(x)$ to evaluate $\frac{2m}{\hbar^2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} V(x)\psi(x)dx$.

9. Now put all the pieces together to form the equation that controls the discontinuity in $d\psi/dx$. Solve this equation for κ.

10. Recall that $\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$. Rewrite your equation for κ as an equation in E and solve that equation for E.

11. How many values of E can give us physically satisfactory bound state solutions?

12. If you have time, normalize $\psi(x)$ and write down the resulting normalized wave function.

13. What is the expectation value of x for this bound state? (Hint: just think about it and write down the answer.) What is the expectation value for x^2? (You may have to actually do some work on that one....)