Exchange “Forces” Tutorial

In this tutorial we will derive some interesting properties of the separation between two identical particles in a quantum system. We will examine a situation in which we have two particles, one of which is in the state ψ_a and the other in the state ψ_b. We will consider three different cases. In the first case the two particles are distinguishable and particle 1 is in state ψ_a, so the wave function can be written:

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2).$$

In the second case the two particles are identical bosons and the wave function is

$$\psi_+(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\psi_a(x_1)\psi_b(x_2) + \psi_b(x_1)\psi_a(x_2) \right).$$

In the third case the two particles are identical fermions and the wave function is

$$\psi_-(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\psi_a(x_1)\psi_b(x_2) - \psi_b(x_1)\psi_a(x_2) \right).$$

In this tutorial we will assume that the single particle states are orthonormal, so for example

$$\int |\psi_a(x)|^2 dx = 1 \qquad \text{and} \qquad \int \psi_a^* \psi_b dx = 0.$$

We will also introduce a shorthand for expectation values. For example, the expectation value of x_1 for the state ψ_b is written as $\langle x \rangle_b$ where

$$\langle x \rangle_b = \int x |\psi_b(x)|^2 dx.$$

Similarly,

$$\langle x \rangle_{ab} = \int x \psi_a^*(x) \psi_b(x) dx.$$

1. Determine the expectation value of the square of the separation between the two particles, $(x_1 - x_2)^2$, for the state ψ. Write your answer in the shorthand notation described above.

2. Determine the expectation value of the square of the separation between the two particles, $(x_1 - x_2)^2$, for the state ψ_+. Write your answer in the shorthand notation described above.

3. Determine the expectation value of the square of the separation between the two particles, $(x_1 - x_2)^2$, for the state ψ_-. Write your answer in the shorthand notation described above.

4. Now compare your answers for the three cases. If the quantity $\langle x \rangle_{ab} = 0$, how do they compare?

5. Note that $\langle x \rangle_{ab}$ will be zero if the wave functions ψ_a and ψ_b have no spatial overlap (in other words, for any x if $\psi_a(x) \neq 0$ then $\psi_b(x) = 0$ and vice versa). Why might we expect the results to be similar in all three cases when this condition holds?

6. What if $\langle x \rangle_{ab} \neq 0$? Which case has the greatest expectation value for the square of the separation? Which case has the smallest expectation value?

7. We can interpret these results by saying that bosons that are brought into close proximity to each other tend to attract each other (so that they will be, on average, closer together than they otherwise would be). Fermions brought close together (so that their wave functions overlap) tend to repel each other (so that they will be, on average, farther apart than they otherwise would be). Distinguishable particles have no tendency to repel or attract in this way. This tendency of bosons to attract each other, and of fermions to repel each other, is sometimes called an "exchange force." Explain why this term is misleading.

8. Imagine you had two varieties of particle which were identical in all ways except one had spin-1 while the other had spin-3/2. Now suppose you try to pack a bunch of the spin-1 particles together as tightly as possible, and you do the same thing for the spin-3/2 particles. Which set of particles will occupy the smallest volume?