Angular Momentum Tutorial

In this activity you will make use of the following commutation relations:

- $[x_i, p_j] = i\hbar\delta_{ij}$ where i and j are indices for x, y, and z,
- $[AB, C] = A[B, C] + [A, C]B$,
- $[A, B \pm C] = [A, B] \pm [A, C]$,
- $[B, A] = -[A, B]$,
- $[A, A] = 0$ for any A.

1. In classical physics angular momentum is defined as: $\vec{L} = \vec{r} \times \vec{p}$. In the space below find the Caresian components of $\vec{L}$ in terms of the components of $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$.

2. Using your results from the previous question as the definition of the operators L_x, L_y, and L_z find the commutator $[L_z, L_x]$.

3. It turns out that we can write the commutators of the angular momentum components in a compact way:
$$[L_i, L_j] = i\hbar L_k,$$
where as long as ijk is a cyclic permutation of xyz. Show that this equation can reproduce your result for $[L_z, L_x]$.

4. $[L_y, L_x] =$ ____________________________ (Hint: work out $[L_x, L_y]$ first.)

5. $[L_x, L_x] =$ ____________________________

6. Compute $[L^2, L_y]$ where $L^2 = L_x^2 + L_y^2 + L_z^2$.

7. Based on the above results can we define a quantum state that is *simultaneously* an eigenstate of L_x and an eigenstate of L_y?

8. Can we define a quantum state that is *simultaneously* an eigenstate of L_x and L^2? (Note: it is easy to show that $[L^2, L_z] = [L^2, L_x] = 0$.)

9. Let's try to find simultaneous eigenstates for L^2 and L_z. So we need functions f that satisfy
$$L^2 f = \lambda f \qquad \text{and} \qquad L_z f = \mu f.$$
To try to find these functions we will start by defining raising and lowering operators (as we did with the harmonic oscillator):
$$L_\pm \equiv L_x \pm i L_y.$$
Find the commutator $[L_z, L_\pm]$.

10. What is the commutator $[L^2, L_\pm]$?

11. Show that if f is an eigenfunction as assumed above, then $L_\pm f$ is also a simultaneous eigenfunction of L^2 and L_z. Determine the eigenvalues of L^2 and L_z for the state $L_\pm f$.

12. So L_+ raises the eigenvalue of L_z by $\hbar$ and L_- lowers that eigenvalue by $\hbar$. Now presumably the largest value of μ can't be larger than $\sqrt{\lambda}$ (otherwise the component L_z is larger than the magnitude of the whole vector!). So there must be a "top" state, f_t, for which:

$$L_z f_t = \hbar\ell f_t, \qquad L^2 f_t = \lambda f_t, \qquad \text{and } L_+ f_t = 0.$$

Use the identity $L^2 = L_- L_+ + L_z^2 + \hbar L_z$ (which we won't prove here) to show that $\lambda = \hbar^2 \ell(\ell+1)$. (Hint: use the identity to rewrite $L^2 f_t = \lambda f_t$.)

13. In the same way μ must be greater than $-\sqrt{\lambda}$, so there must be a bottom state, f_b, with

$$L_z f_b = \hbar\bar{\ell} f_b, \qquad L^2 f_b = \lambda f_t, \qquad \text{and } L_- f_b = 0.$$

Use the identity $L^2 = L_+ L_- + L_z^2 - \hbar L_z$ (easily derived from the previous identity) to show that $\lambda = \hbar^2 \bar{\ell}(\bar{\ell} - 1)$.

14. Solve for $\bar{\ell}$ in terms of ℓ.