# Using Dirac Notation

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1. Consider a 2-D Hilbert space with unit vectors
$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad\qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
Now suppose there is an observable $\hat{A}$ such that $\hat{A} = \alpha|1\rangle\langle 2| + \alpha|2\rangle\langle 1|$. Write the operator $\hat{A}$ as a matrix in the $1-2$ basis.

2. Now we want to find the eigenvalues of this observable. To do this we set $\det\left(\hat{A} - \lambda\hat{I}\right) = 0$, where $\hat{I}$ is the 2 by 2 identity and $\lambda$ is an eigenvalue of the observable $A$. Set up this equation in the space below and then solve for $\lambda$. Note that it should be a quadratic equation in $\lambda$, so you should get two solutions.

3. Next we want to find the eigenstates (or eigenvectors) that go with each eigenvalue. To do this we set up the equation $\left(\hat{A} - \lambda_1\hat{I}\right)|A_1\rangle = 0$, where $\lambda_1$ is the first eigenvalue and
$$|A_1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}.$$
This gives us a system of two equations in the unknowns $a$ and $b$. Solve for $b$ in terms of $a$. Then use the normalization condition ($\langle A_1|A_1\rangle = 1$) to solve for $a$.

4. Repeat the above procedure to find the eigenvector $|A_2\rangle$ associated with the eigenvalue $\lambda_2$.

5. Show that $|A_1\rangle$ and $|A_2\rangle$ are orthogonal.

6. Now let's convert to the $A_i$ basis. We'd like to represent $\hat{A}$ in this new basis. We know that

$$\langle A_i|\hat{A}|A_j\rangle = \sum_{n,m}\langle A_i|n\rangle\langle n|\hat{A}|m\rangle\langle m|A_j\rangle.$$

   This is really just the product of three matrices. We already know the matrix $\langle n|\hat{A}|m\rangle$ (from the first question). In the space below, determine the matrix $\langle m|A_j\rangle$.

7. Now find $\langle A_i|n\rangle$.

8. Now multiply the three matrices (IN THE RIGHT ORDER!) to find $\langle A_i|\hat{A}|A_j\rangle$. Does your result make sense?

9. Let's introduce a new observable $\hat{B} = -i\alpha|1\rangle\langle 2| + i\alpha|2\rangle\langle 1|$. Find the eigenvalues $(b_1, b_2)$ and eigenvectors $(|B_1\rangle, |B_2\rangle)$ of $\hat{B}$ in the 1-2 basis.

10. Show that the two eigenvectors are orthogonal.

11. Now write the operator $\hat{A}$ (from the previous tutorial) in the $|B_i\rangle$ basis.

12. Write the eigenvectors of $\hat{A}$ in the $|B_i\rangle$ basis.

13. Show that the vectors you found in the previous question really are eigenvectors of $\hat{A}$ by multiplying these vectors by the matrix for $\hat{A}$ written in the $B$-basis.