Predictions and Probability

In this activity we will set aside the problem of solving for the wave function of a particle and instead focus on what we can DO with the wave function once we have it. As discussed before, quantum mechanics allows us to make *probabilistic predictions* about the outcome of a measurement. We will examine some forms of probabilistic prediction in this tutorial.

Probabilistic predictions are tricky. For example, if the weather forecast calls for a 60% chance of snow then how can we test the accuracy of this prediction? Not with a single measurement. If it snows or doesn't snow, neither outcome refutes the prediction. So it might seem at first that probabilistic prediction isn't really testable at all.

But we CAN test probabilistic predictions, if we are careful about how we do it. The idea is to create an ensemble of identical systems, and then measure all of these systems. We can use the outcome of these many measurements to test our probabilistic predictions. For example, suppose we could create a million different systems in which the atmospheric conditions are identical. If our prediction calls for a 60% chance of snow, then we can watch all the different systems to see if it snows. If it snows in roughly 600,000 cases and doesn't snow in the other 400,000, then our prediction was accurate.

Quantum mechanical predictions are tested in the same way. We always think of having an ensemble consisting of a large number of systems, all with identical wave functions $\Psi(x,t)$. In this tutorial we will focus on position measurements (or, equivalently, measurements of quantities that are functions of position only). We will consider three different basic probabilistic predictions:

Probability on an Interval The probability that a measurement of x will result in a value on the interval $a < x < b$.

- **Measurement** Measure x in each of the N members of the ensemble. Find the number n of results that are on the interval $a < x < b$. Divide n by N to get the probability P.
- **Quantum Prediction** The probability is given by

$$P = \int_a^b |\Psi(x,t)|^2 dx.$$

Expectation Value The average result of measuring a quantity Q in all members of the ensemble.

- **Measurement** Measure the quantity $\hat{Q}$ in each member of our ensemble, then compute the average $\langle\hat{Q}\rangle$ of all these measurement results.
- **Quantum Prediction** Assuming $\hat{Q} = Q(x)$ is a function of x only, then the *expectation value* of $\hat{Q}$ is given by

$$\langle\hat{Q}\rangle = \int_{-\infty}^{\infty} Q(x)|\Psi(x,t)|^2 dx.$$

Standard Deviation A measurement of the spread of measurement results for the quantity Q.

- **Measurement** Measure $\hat{Q}$ (where $\hat{Q} = Q(x)$ is a function of x only) on half the members of the ensemble, and $\hat{Q}^2$ on the other half. Compute the averages $\langle\hat{Q}\rangle$ and $\langle\hat{Q}^2\rangle$. Compute the standard deviation $\sigma_Q = \langle\hat{Q}^2\rangle - \langle\hat{Q}\rangle^2$.
- **Quantum Prediction** Use the same definition for σ_Q given above, but compute the expectation values using the quantum mechanical formulas.

1. As a first example, let's consider a system with the wave function

$$\Psi(x,t) = \begin{cases} e^{-iEt/\hbar}\left(4-(x/a)^2\right), & |x| \le 2a \\ 0, & |x| > 2a. \end{cases}$$

Find $|\Psi(x,t)|^2$. Is it a function of x and t, or a function of x only?

2. Sketch $|\Psi(x,t)|^2$ as a function of x. Your sketch doesn't have to be perfect, but try to get the basic shape, width, and positioning correct.

3. Normalize $\Psi(x,t)$.

4. Find the probability that a measurement of x will yield a result in the interval $-a < x < a$.

5. Compute the expectation value $\langle x \rangle$.

6. Compute $\langle x^2 \rangle$.

7. Compute σ_x.

8. Compare your calculation results to the sketch you made earlier. Explain how your calculations agree, or fail to agree, with your sketch.