Spin Tutorial

In this tutorial we will work out the mathematical theory for spin-1/2 particles. So we will only consider $s = 1/2$, in which case we have $m_s = \pm 1/2$. We have no differential equation to solve, so we must work out the theory from the algebraic properties of the operators and their eigenstates. We begin by defining our basis vectors to be the simultaneous eigenstates of S^2 and S_z:

$$|s = 1/2, m = 1/2\rangle = \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |s = 1/2, m = -1/2\rangle = \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Spin states, which in the case of $s = 1/2$ can be written as 2-vectors, are referred to as "spinors". The eigenstates of S^2 and S_z are *eigenspinors*.

1. Determine the matrix for S^2 in the $\chi_\pm$ basis. Recall that $S^2\chi_\pm = \hbar^2(1/2)(1/2 + 1)\chi_\pm = (3\hbar^2/4)\chi_\pm$.

2. Determine the matrix for S_z in the $\chi_\pm$ basis. Recall that $S_z\chi_+ = (\hbar/2)\chi_+$ and $S_z\chi_- = -(\hbar/2)\chi_-$.

3. We can define raising and lowering operators such that $S_\pm|sm\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)}|s(m \pm 1)\rangle$. Use this formula to evaluate $S_+\chi_+$, $S_+\chi_-$, $S_-\chi_+$, and $S_-\chi_-$.

4. Determine the matrices for the raising and lowering operators ($S_\pm$) in the $\chi_\pm$ basis.

5. Use your results for the raising and lowering operators to construct matrices for S_x and S_y in the $\chi_\pm$ basis. Recall that $S_\pm = S_x \pm iS_y$.

6. We can define a set of three matrices known as the *Pauli matrices*:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \text{and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $S_x = (\hbar/2)\sigma_x$, $S_y = (\hbar/2)\sigma_y$, and $S_z = (\hbar/2)\sigma_z$.

7. Are the Pauli matrices Hermitian? Are S_x, S_y, and S_z Hermitian?

8. Find the expectation value for S_y for the state $\psi = a\chi_+ + b\chi_-$.

9. Find the eigenvalues and eigenspinors of S_x in the $\chi_\pm$ basis.

10. Rewrite the state $\psi = a\chi_+ + b\chi_-$ in the basis of eigenstates of S_x (which we can denote $\chi_\pm^{(x)}$). Make use of the identity operator written in the new basis:

$$\hat{1} = |\chi_+^{(x)}\rangle\langle\chi_+^{(x)}| + |\chi_-^{(x)}\rangle\langle\chi_-^{(x)}|.$$

11. If we were to measure S_x for the state ψ, what is the probability that we would get $\hbar/2$? What is the probability that we would get $-\hbar/2$?

12. Consider the state $\psi = A\,(3\chi_+ + 4\chi_-)$. Normalize this state. What is the probability that we measure $S_z = \hbar/2$ for this state? What is the probability that we measure $S_z = -\hbar/2$? What is the probability that we measure $S_x = \hbar/2$? What is the probability that we measure $S_x = -\hbar/2$?