Harmonic Oscillator: Commutators and Ladder Operators

In this tutorial you will construct and use the energy eigenstates for the harmonic oscillator that we found last time. We will also introduce the *commutator* of two operators, and use this notion to define raising and lowering (ladder) operators for the harmonic oscillator.

1. Last time we solved the EEP for the harmonic oscillator. The (normalized) solution was

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

   where $\xi = \sqrt{m\omega/\hbar}x$ and the H_n are Hermite polynomials given in Table 2.1 on p. 56 of the text. The energy eigenvalues are $E_n = \hbar\omega(n + 1/2)$. Construct the expressions for $\psi_0(x)$ and $\psi_1(x)$ in the space below.

2. We will see later in the course that energy eigenstates are *orthonormal*. This means that the *inner product*

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi_m(x)dx = \delta_{nm}$$

   where δ_{nm} is the Kronecker delta. Use the functions for ψ_0 and ψ_1 to show that this works for $n = 0$ and $m = 1$ in the harmonic oscillator.

3. Now show that this also works for $n = 0$ and $m = 0$.

4. Suppose you have an ensemble of particles all with the same wave function Ψ. You measure the energy of these particles and find that 40% of the time you get $E = \hbar\omega/2$, while the remaining 60% of the time you get $E = 3\hbar\omega/2$. Write one possible wave function $\Psi(x, t)$ for all t in the space below. (Note: you may use ψ_0 and ψ_1 as shorthand for the expressions you wrote above. The wave function you write down will be one of an infinite variety of possibilities with different relative phases.)

5. Now we are going to switch gears and talk about operators. Consider the operator x. It acts on a function $f(x)$ to produce the new function $xf(x)$. Likewise our momentum operator $\hat{p} = -i\hbar(d/dx)$ acts on a function $f(x)$ to produce the new function $-i\hbar f'(x)$. We define the *commutator* of two operators as the difference between the results of applying the operators in opposite order:

$$[\hat{O}, \hat{Q}] = \hat{O}\hat{Q} - \hat{Q}\hat{O}.$$

Use the definition of the commutator to show that

$$[\hat{Q}, \hat{O}] = -[\hat{O}, \hat{Q}].$$

6. Evaluate the commutator $[x, p]$ by applying the commutator to an (arbitrary) test function $f(x)$. (In other words, try to work out what $[x, p]f(x)$ is, and see if you can factor the $f(x)$ out of the end result.)

7. Now let's define some new operators, the "raising" operator (a_+) and the "lowering" operator (a_-), which are defined as follows:

$$a_\pm = (2\hbar m\omega)^{-1/2}(\mp ip + m\omega x).$$

We'll see why these operators have these names later. For now, let's try to evaluate the commutator $[a_-, a_+]$. (Hint: you should be able to rewrite this commutator in terms of the *fundamental commutator* $[x, p]$, then use your result for the fundamental commutator found above.)

8. Show that $a_- a_+ = \frac{H}{\hbar\omega} + \frac{1}{2}$ where $H = p^2/(2m) + (1/2)m\omega^2 x^2$ is the Hamiltonian (total energy) operator for the harmonic oscillator.

9. So $H = \hbar\omega(a_- a_+ - 1/2)$. Show, using your result for $[a_-, a_+]$, that $H = \hbar\omega(a_+ a_- + 1/2)$.

10. Challenge: Verify that Rodgrigues formula

$$H_n(\xi) = (-1)^n e^{\xi^2} \left(\frac{d}{d\xi}\right)^n e^{-\xi^2}$$

reproduces the first 4 Hermite polynomials in Table 2.1 (p. 56). If you have time, do the same thing for $H_n(\xi) = (d/dz)^n \exp(-z^2 + 2z\xi)|_{z=0}$.