Statistical Interpretation Tutorial

We have seen that the absolute square of the wave function $\Psi(x,t)$ can be interpreted as the probability density for the location of the particle at time t. We have also seen that a particle's quantum state can be represented as a linear combination of eigenstates of a physical observable $\hat{Q}$:

$$|\Psi\rangle = \sum_n c_n |n\rangle$$

where $\hat{Q}|n\rangle = q_n|n\rangle$ and $|c_n|^2$ represents the probability to find the particle in the eigenstate $|n\rangle$ (and thus to measure the value q_n for the observable Q.

1. Calculate the expectation value of Q for the state $|\Psi\rangle$ given above.

2. Consider a particle in the infinite square well system (with energy eigenvalues $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$). Suppose the state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_n c_n |n\rangle$$

where $|n\rangle$ is an energy eigenstate with eigenvalue E_n and $c_n = A/n^2$. Find the appropriate value for A to normalize this state. (Hint: $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.)

3. Now determine the expectation value of energy for the state described in the previous problem. (Hint: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.)

4. We have seen that we can represent a quantum state in a basis of eigenstates for a physical observable that has a continuous spectrum. In fact, this is exactly what we are doing when we write the wave function $\Psi(x,t)$. In that case we are using a basis of position eigenstates. We can also write the quantum state in a basis of momentum eigenstates ($\hat{p}|p\rangle = p|p\rangle$):

$$|\Psi\rangle = \int_{-\infty}^{\infty} c(p,t)|p\rangle dp$$

where now $|c(p,t)|^2$ can be interpreted as a probability density for the momentum of the particle at time t. Show that $c(p,t) = \langle p|\Psi\rangle$. (Hint: use the Dirac orthonormality of the momentum eigenstates.)

5. The quantity $c(p,t)$ turns out to be pretty useful, so we give it a special symbol $c(p,t) = \Phi(p,t)$ and call it the*momentum-space wave function* of the particle. Show that

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,t)dx$$

by first writing the state $|\Psi\rangle$ in the position basis, inserting the identity operator written in the momentum basis ($\hat{I} = \int_{-\infty}^{\infty} |p\rangle\langle p|dp$). Note that the position-space wave function for a momentum eigenstate is

$$\langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}.$$

6. Consider the ground state of the harmonic oscillator

$$\Psi(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/(2\hbar)} e^{-i\omega t/2}.$$

Find the momentum-space wave function for this state. (Hint: $\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = \sqrt{\pi/a}\; e^{b^2/(4a)}$.)

7. What is the classically allowed range of momenta for this particle? (Hint: the particle's total energy is $\hbar\omega/2$.

8. What is the probability to find this particle OUTSIDE of the classical range?