“How Big Is The Cantor Set?”

A Debate Concerning the Mathematical Notion of Size

BETWEEN:

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&

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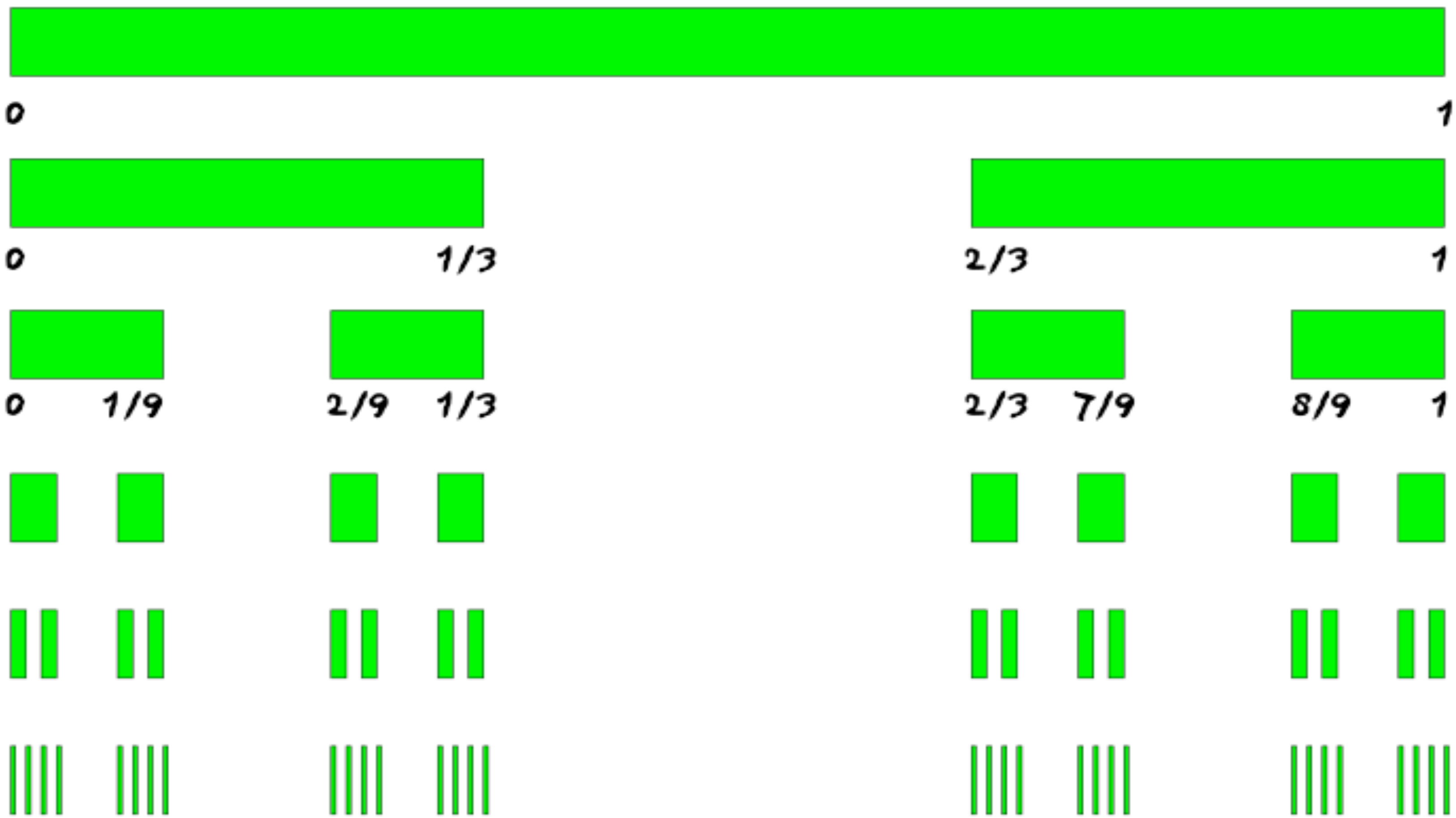
The Cantor Set

Why it is VERY small

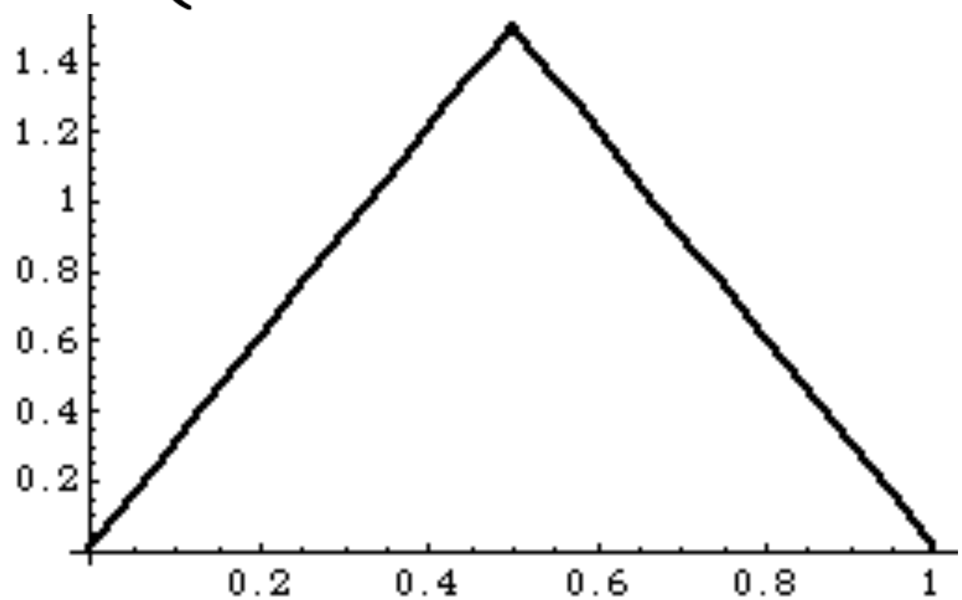
The Cantor Set



The Tent Map

- The points in the Cantor set are mapped to other points in the Cantor set by the Tent Map

$$f(x) = \begin{cases} 3x, & \text{if } 0 \le x \le 0.5 \\ 3(1-x), & \text{if } 0.5 < x \le 1 \end{cases}$$



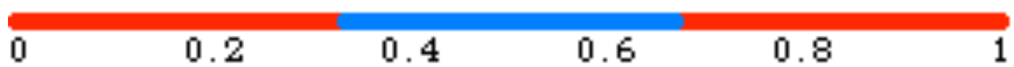
Points in the Cantor Set

- Dr. Lazlo tells us that the following points are in the Cantor set: 0, 1/9, 2/9, 1/3, 2/3, 7/9, 8/9, and 1.
- Let's examine what the Tent Map does to each of these points:

$$f(x) = \begin{cases} 3x, & \text{if } 0 \le x \le 0.5 \\ 3(1-x), & \text{if } 0.5 < x \le 1 \end{cases}$$

Tent Map and Cantor Set

- All points not in the Cantor set are eventually mapped out of the interval [0,1] by this map.
- So if we apply the map many times, what is left should be the Cantor set!



Cantor Set Movie

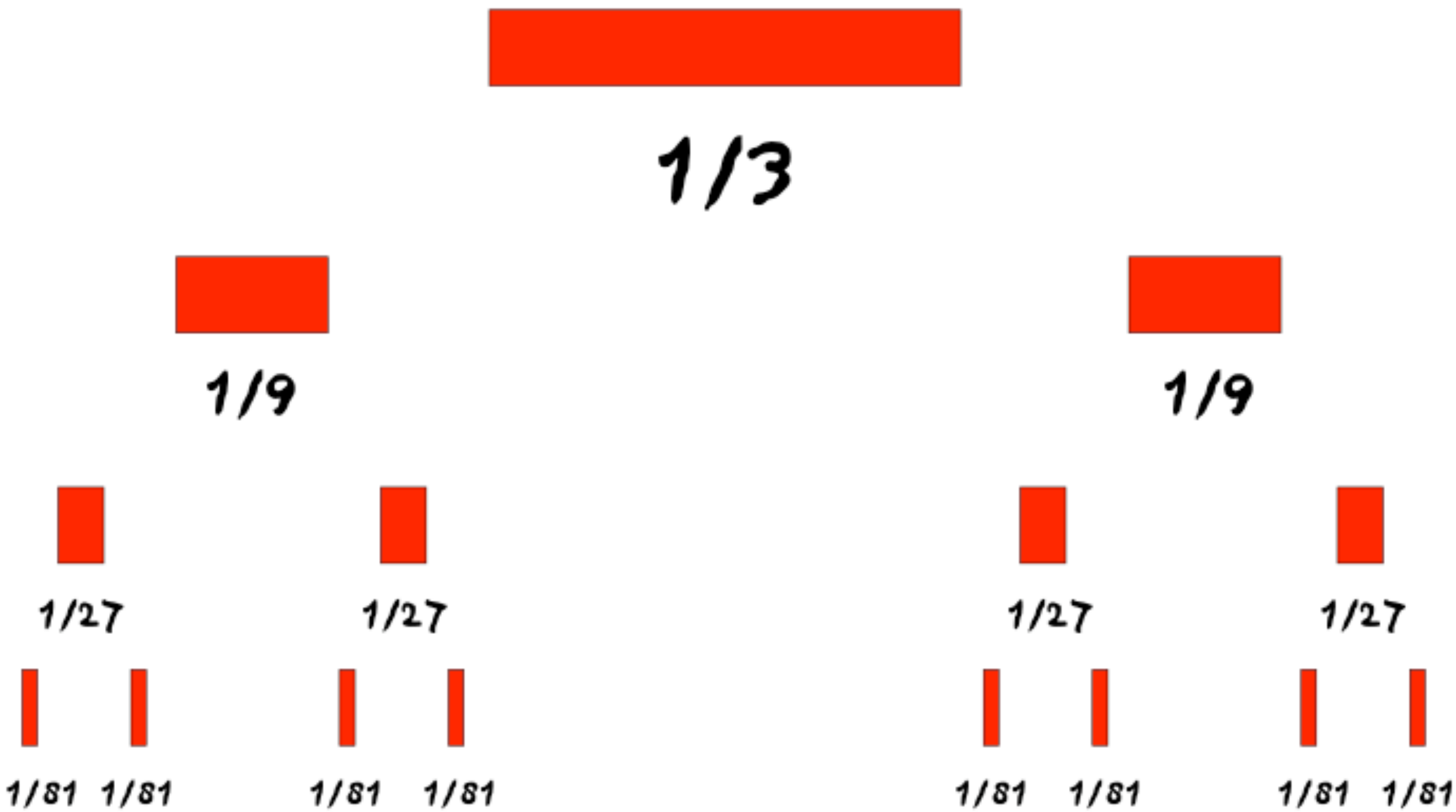
The Length of the Cantor Set

- Let's find the length of the interval that remains after each step of the construction of the Cantor set.
- After step 1: $L_1 = 1/3 + 1/3 = 2/3$
- After step 2: $L_2 = 1/9 + 1/9 + 1/9 + 1/9 = 4/9$
- At each step we take away 1/3 of what remains, so the length is multiplied by 2/3.
- After the n^{th} step: $L_n = (2/3)^n$

The FINAL Length

- We only have the Cantor set after performing an infinite number of the these steps.
- To find the length of the Cantor set, we see what happens to L_n as $n \Rightarrow \infty$.
- As $n \Rightarrow \infty$, $(2/3)^n$ becomes 0!
- So the Cantor set has no length. That's pretty small if you ask me…

Not The Cantor Set



The Length of What's Not in the Cantor Set

- In the first step we remove 1/3, so $L_1 = 1/3$.
- In the second step we remove two sections of length 1/9. We must **add** these lengths to L_1 to find the total length that has been removed. So $L_2 = 1/3 + 2/9 = 5/9 = 0.55555...$
- Next we remove four sections of length 1/27, so $L_3 = 5/9 + 4/27 = 19/27 = 0.703703\ldots$
- $L_4 = 19/27 + 8/81 = 65/81 = 0.802469\ldots$

Total Length of What's Not in the Cantor Set

- The pattern we find is that $L_n = 1-(2/3)^n$.
- As $n \Rightarrow \infty$, $1-(2/3)^n$ goes to 1!
- Therefore to create the Cantor Set we must remove a series of intervals whose total length is equal to the length of what we started with.
- So what is left (the Cantor Set itself) must have no length at all!

Conclusion

- To construct the Cantor Set we start with an interval of length 1 and remove pieces whose total length is also 1.
- We have shown, in several different ways, that the length of the Cantor Set is 0.
- If something with length 0 isn't small, then what is?