A COMPUTATIONAL APPROACH TO TEACHING CONSERVATIVE CHAOS

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Gordon Research Conference on Physics Research and Education
Mount Holyoke College, South Hadley, MA
June 13-18, 2004

ABSTRACT: Studying chaos in conservative systems helps to illustrate many important features of classical nonlinear dynamics and highlights some important issues in quantum-classical correspondence, but few undergraduates physics majors are exposed to this fascinating subject. This poster will describe one way to incorporate the study of conservative chaos into an upper-level classical mechanics course. The focus is on using a general-purpose computing program such as *Mathematica* to illustrate many of the important features of chaos in two-dimensional Hamiltonian maps. This approach helps students to quickly acquire the tools they need to carry out their own studies of chaotic systems.

The Goals

1) To get students in an upper-level classical mechanics sequence excited about classical mechanics by presenting modern material on chaotic systems.
2) To teach students the main concepts that are important for understanding chaotic motion in conservative (Hamiltonian) systems.
3) To help students visualize various aspects of chaotic motion.
4) To encourage students to develop proficiency with general-purpose computation software such as *Mathematica* or *Maple*.
5) To provide students with the tools required to carry out independent research on two-dimensional, area-preserving, chaotic maps.

The Basic Approach

The computational approach to teaching conservative chaos presented here consists of four basic steps:

1) The instructor uses a general-purpose computing program (such as *Mathematica* or *Maple*) to prepare several figures that serve to illustrate important concepts in conservative chaos. These figures help students visualize various aspects of motion in a particular system, which is a 2-D, area-preserving, chaotic map. Such a map serves as a model for a conservative classical system.
2) The instructor presents lectures on these concepts, using the figures to illustrate important points.
3) Students are given the code used to create the figures, along with an explanation of what the code does (and possibly how the code works). They are then asked to create similar figures using different parameter values. This requires only very small modifications of the code.
4) Once the entire unit on conservative chaos has been completed, students are asked to carry out an independent research project (individually or in small groups) in which they will study a 2-D, area-preserving, chaotic map that is different from the one presented in the lectures and homework assignments.

The Standard Map

The map we choose for illustrating the important ideas of conservative chaos is the standard map[1]. The standard map displays many of the features that are characteristic of chaos in conservative systems. The equation of the map is

$$\begin{aligned} r_{n+1} &= r_n - \frac{K}{2\pi}\sin(2\pi\theta_n), && \text{mod } 1 \\ \theta_{n+1} &= \theta_n + r_{n+1}, && \text{mod } 1 \end{aligned}$$

where r and θ are dimensionless phase space variables. Both coordinates are periodic with period 1, so the motion actually takes place on a 2-torus. You can think of r as an action variable and θ as the corresponding angle, or it might be easier to think of r and θ as dimensionless polar coordinates. K is a parameter that controls the nonlinearity of the system. For K = 0 the map is integrable.

The remainder of this poster presents several figures that illustrate different aspects of the dynamics of the standard map. A brief explanation accompanies each figure, but more detailed explanations can be found by visiting the author's website (see the section titled The Website at the end of the poster) or reading one of the recommended texts[2]. In addition, all of the *Mathematica* code used to create these figures is available on the website.

Surfaces of Section

Figure 1 shows the surface of section for the standard map with K = 0.8. Surfaces of section illustrate the overall dynamics of a system. To construct a surface of section, simply choose several initial conditions in various regions of the phase space. For each initial condition, iterate the mapping function many times and plot the resulting points. Surfaces of section illustrate many important phase space features found in conservative systems. Continuous lines that run from left-to-right are KAM tori and represent integrable motion. Elliptical curves are nonlinear resonances, in which the motion of trajectories near a stable periodic point become phase-locked. Regions of the phase space that show a disorganized scatter of points are chaotic. These regions of chaos form when nonlinear resonances overlap.

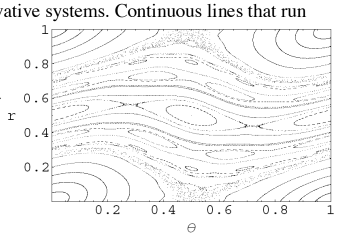

Figure 1

Individual Trajectories

Individual trajectories can also be useful for illustrating certain aspects of the map's dynamics. For example, the chaotic trajectory shown in **Figure 2** appears to be blocked from moving into the middle-third of the phase space. Chaotic trajectories can be blocked in this way by KAM tori. As K increases the KAM tori breakup and chaotic trajectories become free to wander throughout the phase space. In fact, the trajectory shown in **Figure 2** is only partially blocked and it will enter the apparently forbidden region after a sufficiently large number of map iterations.

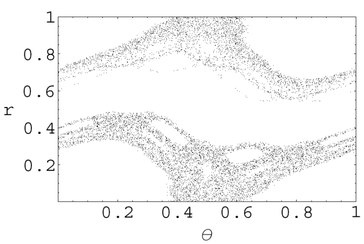

Figure 2

Area-Preservation and Liouville Flow

A collection of trajectories in a conservative system will occupy a constant volume of phase space as the trajectories move around under the dynamics of the system. This is known as Liouville's Theorem. This property of conservative systems can be illustrated in the standard map by applying the map repeatedly to a group of points in the phase space and examining the resulting distribution. **Figure 3** illustrates the flow of points for the standard map with K = 1.5. Each iteration of the map causes the distribution to stretch and bend, but the area occupied by the distribution remains constant.

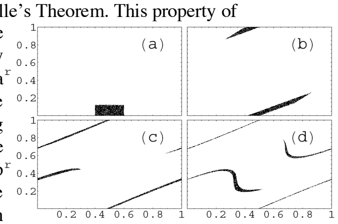

Figure 3

Motion Near an Unstable Fixed Point

Unstable fixed (or periodic) points lie at the heart of the chaotic regions of phase space in conservative systems. Near a typical unstable fixed point, trajectories move in toward the fixed point along one line (called the **stable direction**) and then out away from the fixed point along another line (called the **unstable direction**). Trajectories that start off near the fixed point will move away at an exponentially increasing rate as the map is iterated. This exponential divergence is the essence of chaos.

Figure 4 shows the **stable** and **unstable** directions of the unstable fixed point located at (0.5,0) for the standard map with K = 1.5. These lines are constructed by starting with a group of points in the vicinity of the fixed point. These points will move out along the **unstable** direction under application of the forward map (forming the **black** line) and out along the **stable** direction under application of the inverse map (forming the **red** line). The arrows in **Figure 5** indicate the direction that points will move under application of the map. Note the correspondence between **Figure 4** and **Figure 5**.

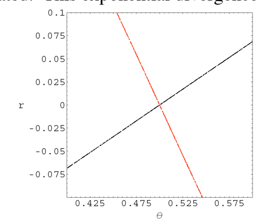

Figure 4

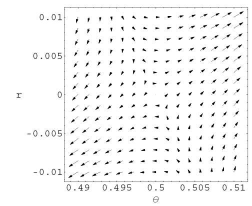

Figure 5

The **stable** and **unstable** directions can be extended to form **stable** and **unstable** manifolds. The **stable** (**unstable**) manifold is the set of all points that eventually move into the fixed point under the application of the forward (inverse) map. In a chaotic system these manifolds form a complicated structure called a homoclinic tangle, a portion of which is shown in **Figure 6**. The complexity of the homoclinic tangle is responsible for the seemingly random motion of trajectories in the chaotic region of the phase space.

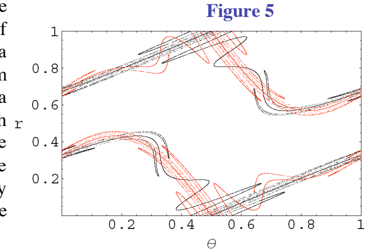

Figure 6

The Website

For more information visit the author's website
http://fsweb.berry.edu/academic/mans/ttimberlake/cc/
which contains an article detailing this approach to teaching conservative chaos, the *Mathematica* code used to generate the figures, animations illustrating these concepts, lecture slides on this material, sample homework assignments, and samples of student work related to this material.

References

[1] Boris V. Chirikov, Phys. Rep. **52**, 263 (1979).
[2] Recommended texts: *Analytical Mechanics* by Hand and Finch, *Chaos and Nonlinear Dynamics* by Hilborn, *Chaos and Integrability in Nonlinear Dynamics* by Tabor.