Is There Quantum Chaos in the Prime Numbers?

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Could the primes be energy eigenvalues of a quantum system?

- For a system with d degrees of freedom, the energy eigenvalues can grow no faster than

$$E_n \approx n^{2/d}$$

- According to the Prime Number Theorem primes grow like

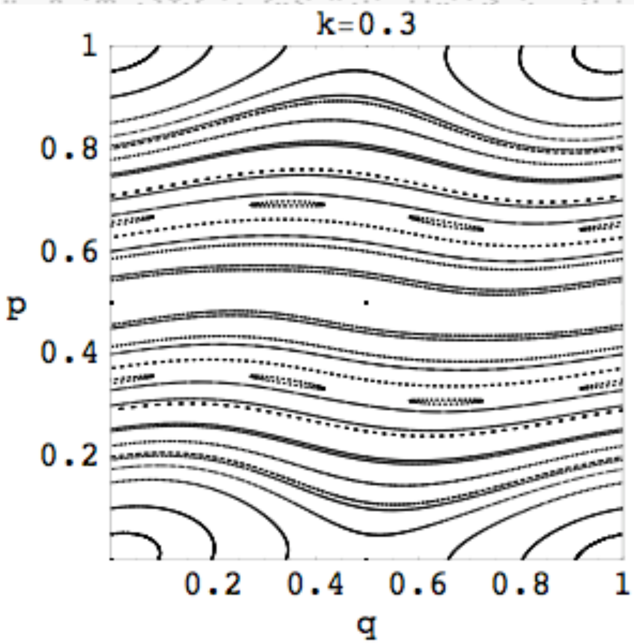
$$p_n \approx n \log n$$

- So the primes *could* be eigenvalues of a 1D system.

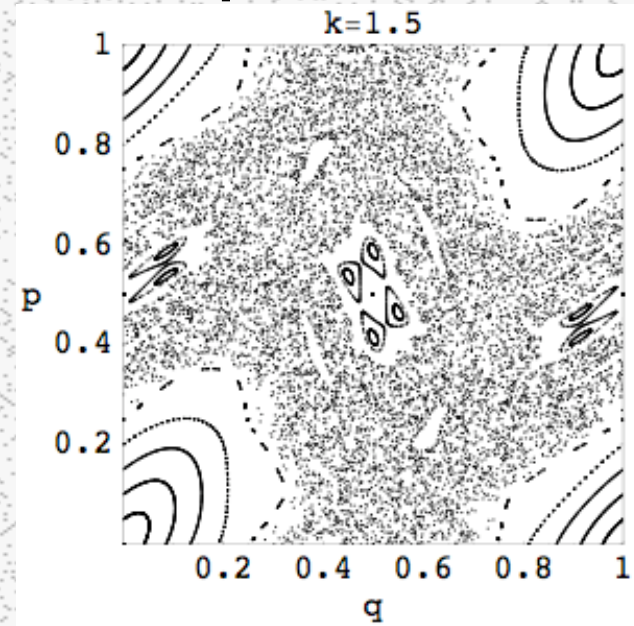
Eigenvalue Statistics

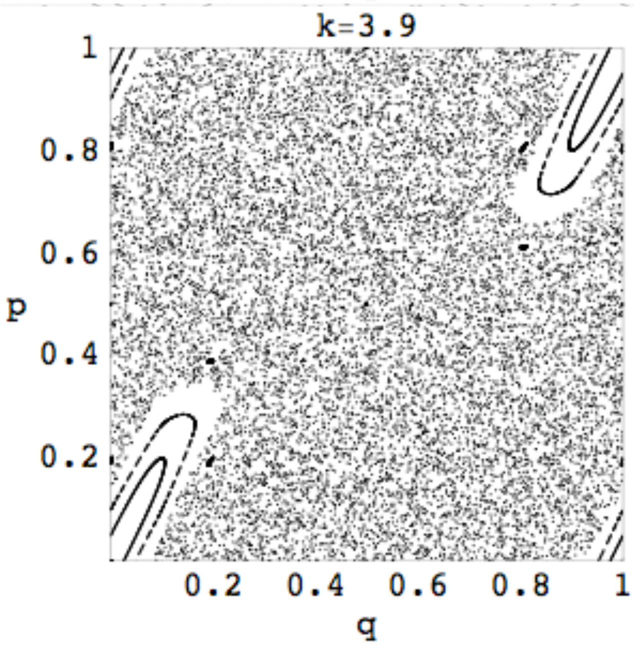
- Key result of “quantum chaos” research is that statistical properties of quantum eigenvalues are related to the dynamics of the classical version of the system.
- Regular dynamics: Poisson statistics.
- Chaotic dynamics: Random matrix statistics (GOE, GUE, GSE).
- Statistical measures: nearest neighbor spacing distribution (NNSD), variance of number of eigenvalues in intervals of length L

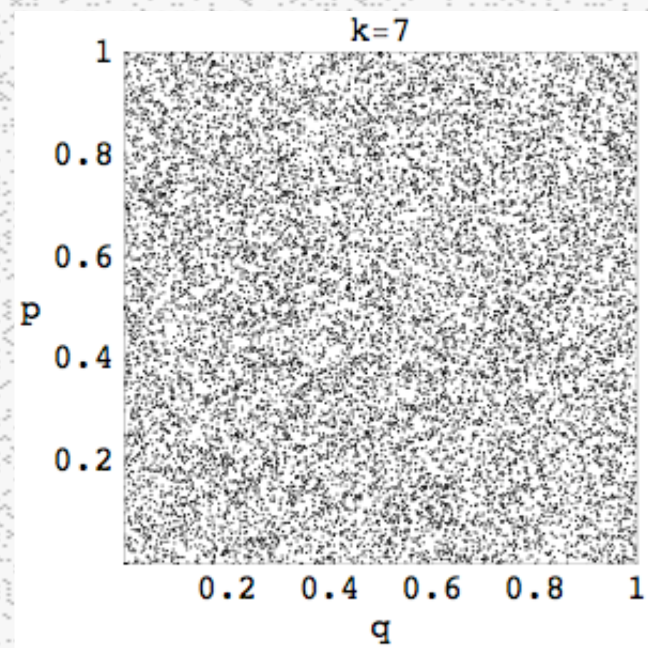
Ex: Classical Standard Map



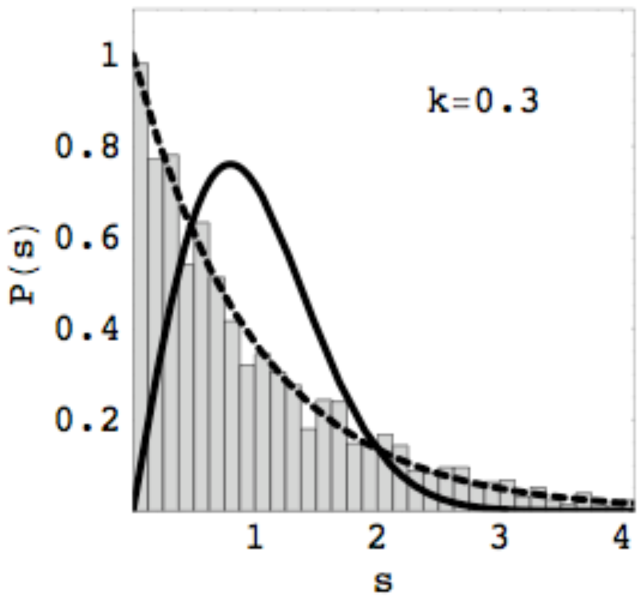
Poincaré Sections of classical motion for various values of the nonlinearity parameter *k*.

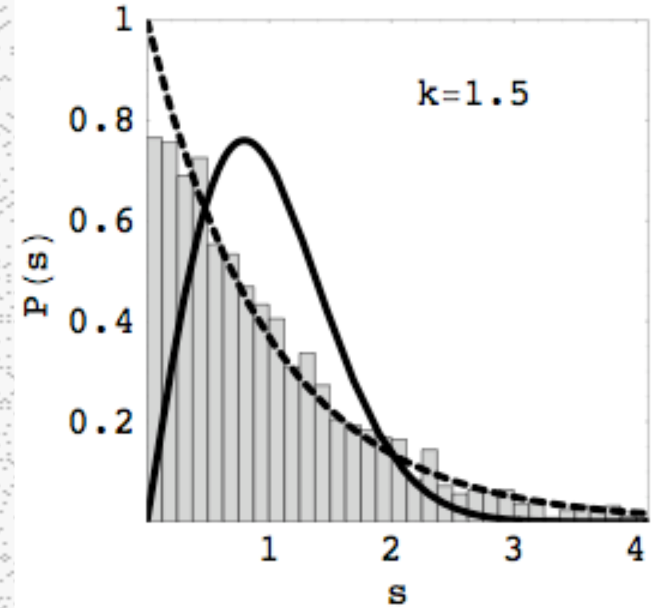




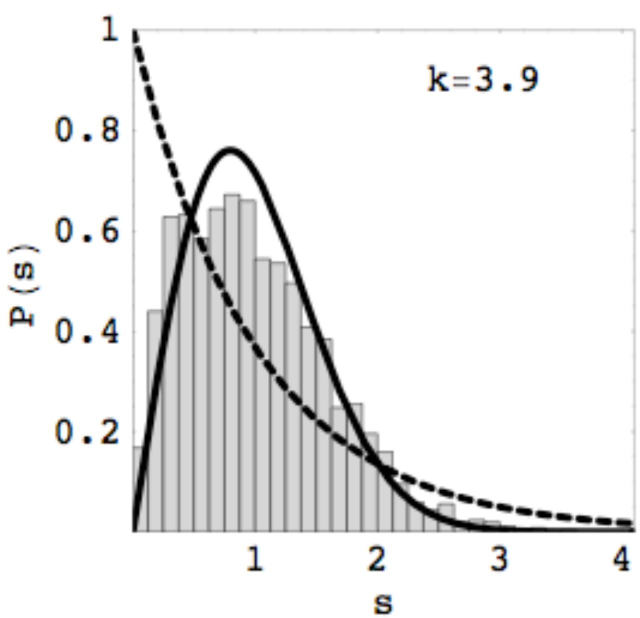


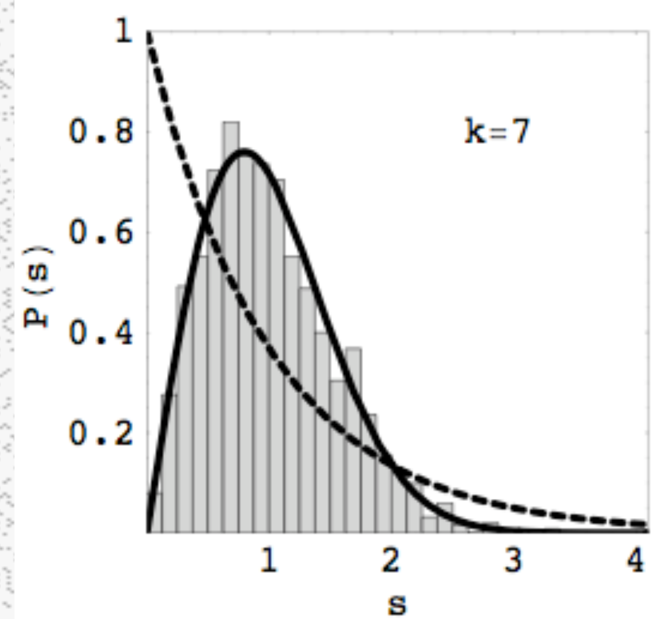
Ex: Quantum Standard Map





Histograms of spacings between consecutive eigenvalues, compared to Poisson (dashed) and GOE (solid) curves.





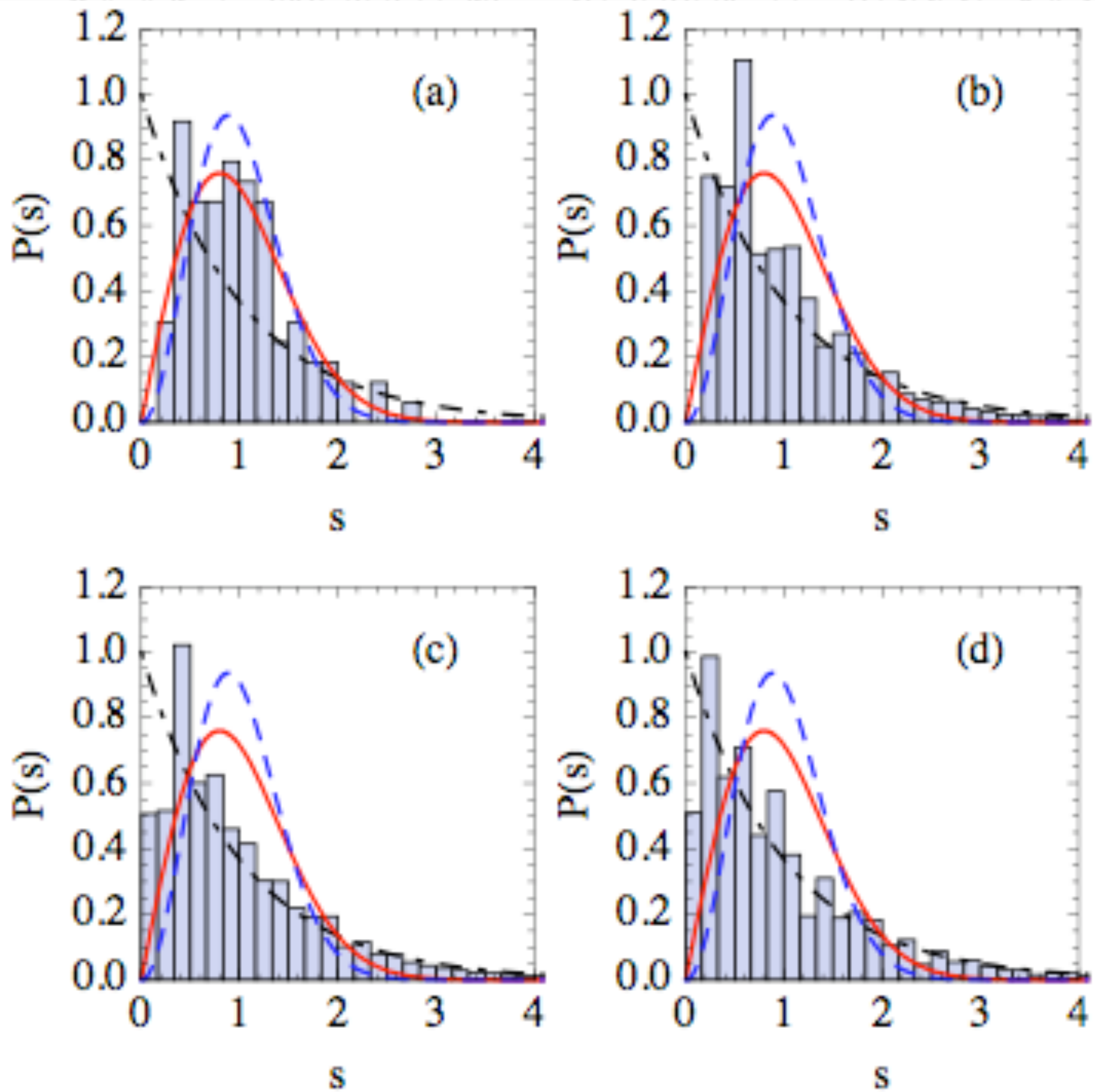
Prime Statistics

- Before computing statistics primes must be *unfolded*: $$\epsilon_n = \mathrm{Li}(\mathrm{p_n}) \approx \mathrm{p_n}/\log \mathrm{p_n}$$
- Two previous studies of nearest neighbor spacing distribution (NNSD):
  - Porter: primes follow Poisson statistics
  - Liboff and Wong: primes follow GOE statistics (no unfolding, poor fit)
- We aim to clarify these results and examine other statistical measures.

NNSD of the primes

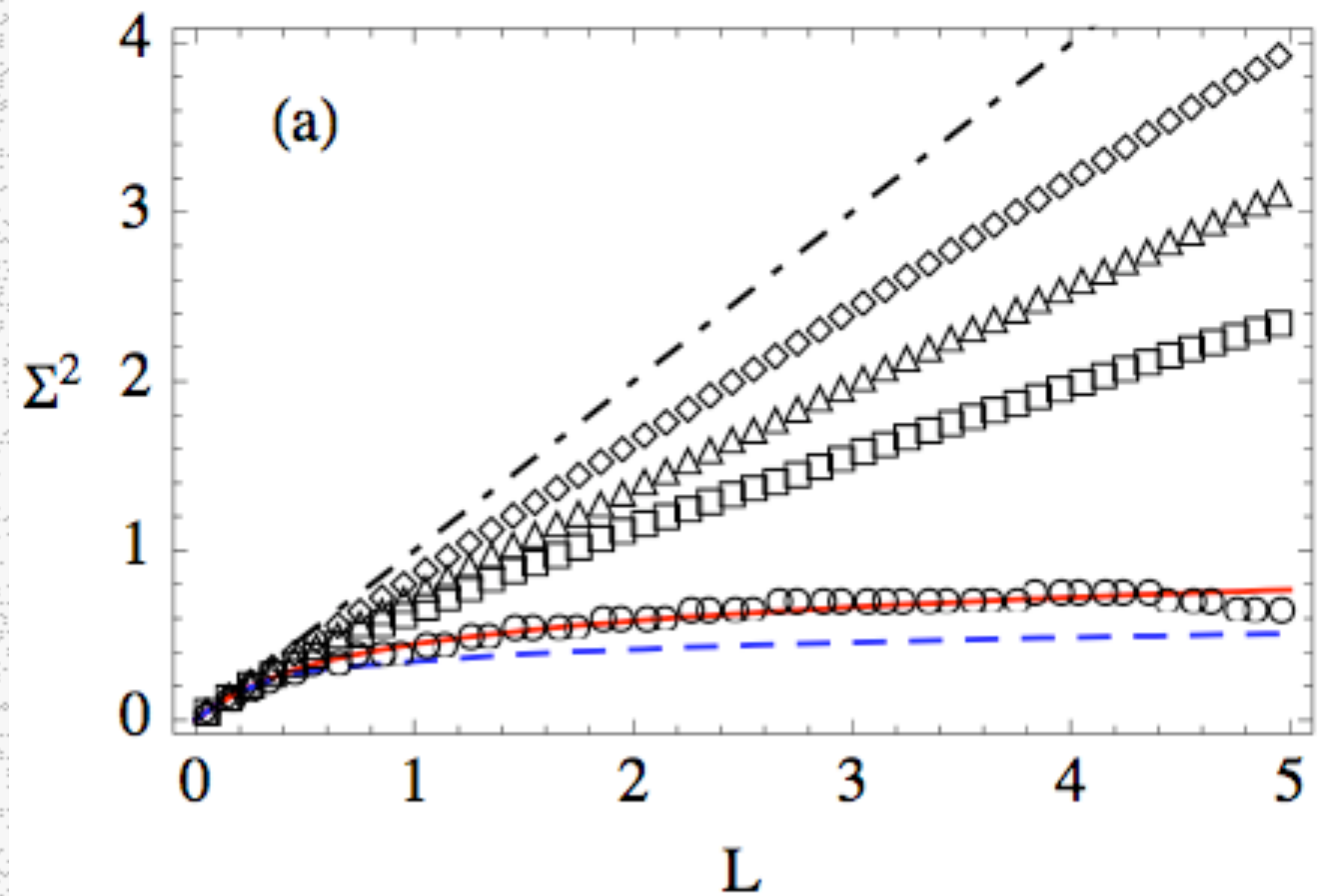
Histograms for the
(a) first 100 primes,
(b) first 10^4 primes,
(c) first 10^6 primes,
(d) first 10^6 primes after the 10^{12}th prime.

Also shown are curves for Poisson (dot-dashed), GOE (red), and GUE (blue dashed) statistics.

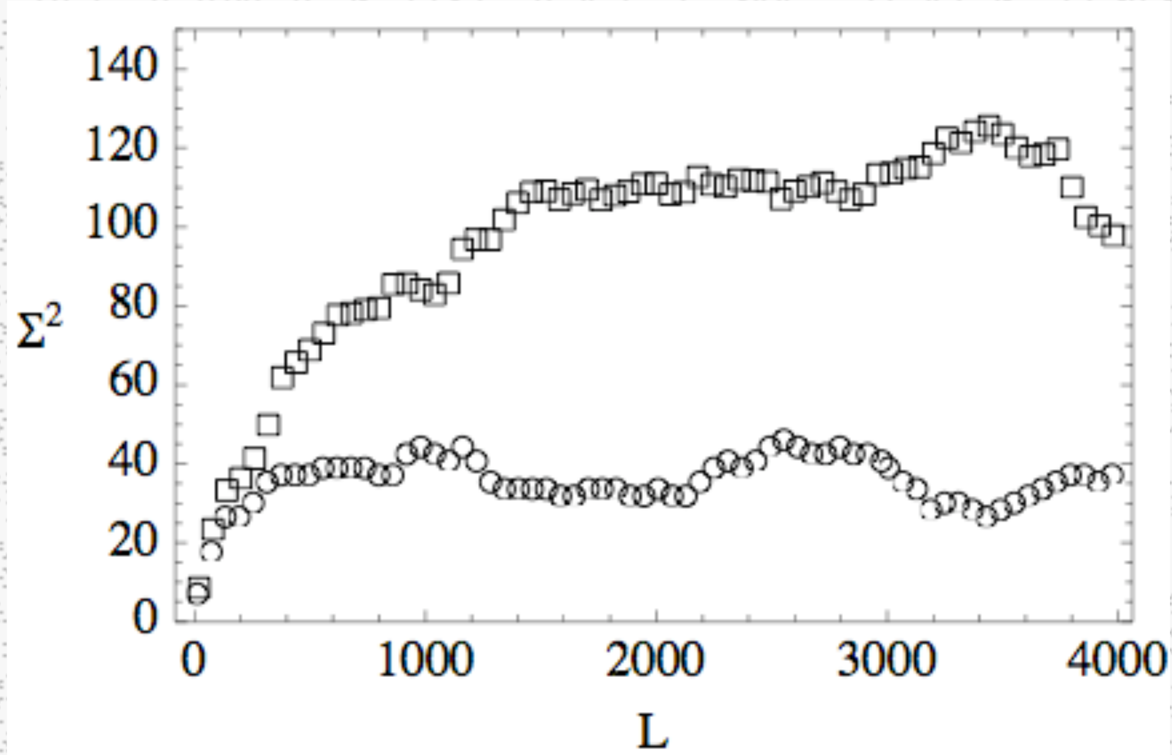


Number Variance of Primes

Number variance for first 100 primes (circles), first 10^4 primes (squares), first 10^6 primes (triangles), first 10^6 primes after the 10^{12}th prime (diamonds), Poisson statistics (dot-dashed), GOE statistics (red), and GUE statistics (blue dashed).



Saturation of Number Variance



- Number variance saturates at a length L that depends on the energy scale of eigenvalues in the sequence.
- This is common for quantum eigenvalues sequences.

Interpretations

- Prime numbers *look like* eigenalues of a quantum system with classical dynamics that is chaotic at low energies and regular at high energies.
- But they can only be eigenvalues of a 1D system (which can't be chaotic).
- Some traces of random matrix statistics have been seen in regular systems (but not 1D systems).
- Driven systems?

References

- T. Timberlake and J. Tucker, arxiv:0708.2567.
- G. Mussardo, arxiv:cond-mat/9712010.
- C. E. Porter, *Statistical Theories of Spectra: Fluctuations* (Academic Press, 1965), pp. 2-87.
- R. Liboff and M. Wong, Int. J. Th. Phys. **37**, 3109 (1998).