Wave Packet Revivals in the Asymmetric Infinite Square Well.

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ABSTRACT: Wave packets in an infinite square well (ISW) experience perfect quantum revivals at periodic intervals. In the asymmetric infinite square well (AISW) a potential step within the well breaks the symmetry of the ISW, perturbing the energy eigenvalues and preventing perfect revivals. Second-order perturbation theory for the AISW shows that the even-numbered eigenstates and the odd-numbered eigenstates should have different revival times. Therefore a wave packet composed of only even-numbered eigenstates should experience partial revivals at a different time than a wave packet composed of only odd-numbered eigenstates. We examine the numerically calculated auto-correlation function of even and odd sub wave packets. We find clear evidence for different revival times within a certain parameter regime. We also find that this difference in even and odd revival times is a leading cause in the poor revivals of the AISW.

Introduction

In the infinite square well (ISW),

\[ V(x) = \begin{cases} \infty, & x < -a \\ 0, & -a < x < a \\ \infty, & x > a \end{cases} \]

a wavepacket will have a periodic motion. It will initially disperse but after a fixed period (known as the revival time) it will return to its initial state. So, for a green wave packet \( \Psi(0|x) \) in the ISW, we know that if \( n \) is an integer and \( \alpha \), the revival time, then

\[ |\Psi(x, t)|^2 = |\Psi(x, n\tau_{rev})|^2. \]

This effect is referred to as a wave packet revival. The ISW experiences perfect wave packet revivals. The revival time for a wave packet peaked on the nth eigenstate in any system is given by

\[ \tau_{rev} = 4\pi \left( \frac{d^2 E_n}{dx^2} \right)^{-1}. \]

For the ISW we have

\[ \tau_{rev} = \frac{\pi a^2}{2h}. \]

Adding a small step in potential halfway through the ISW results in the asymmetric infinite square well,

\[ V(x) = \begin{cases} \infty, & x < -a \\ 0, & -a < x < a \\ \infty, & x > a \end{cases} \]

Wave packets in the AISW do not have perfect revivals. We used perturbation theory and numerical calculations to explain why this potential step causes the perfect wave packet revivals of the ISW to become the imperfect revivals of the AISW.

Perturbation Theory Solutions for the AISW

Perturbation theory can be used to find the energy corrections for the AISW. First-order time independent perturbation theory gives the following formula for the energies of the AISW:

\[ E_n \approx \frac{\pi^2 \hbar^2}{2m(2a)^2} + V_0 \frac{1}{2}. \]

To first order the energies of the AISW are just the ISW energies plus a constant. This will change the overall phase of the wave functions in the AISW, but will not change any of the revival properties. So first order perturbation theory predicts perfect revivals. Examining the second order energy corrections, however, sheds more light on the situation. Assuming that we are dealing with wave packets composed of high energy (large \( n \)) states, the energies of the AISW to second order are

\[ E_n \approx \frac{\pi^2 \hbar^2}{2m(2a)^2} + \frac{V_0}{2} + \frac{\gamma n^2}{8\pi^2 \hbar^2}, \]

where

\[ \gamma \approx 3. \]

Second order perturbation theory predicts a different revival time for even-numbered eigenstates compared to odd-numbered eigenstates. This means that a wave packet composed of only even-numbered states will have a different revival time than a wave packet composed of only odd-numbered states. Values mean that a wave packet composed of both even-numbered and odd-numbered states may experience poor revivals because the even-numbered eigenstates are out of phase with the odd-numbered eigenstates. This is when the "odd" part of the wave packet is ready to revive, the "odd" part isn’t and vice versa. For specific parameter values we find clear evidence for this effect.

Wave Packet

We examined revivals in the AISW for a Gaussian wave packet with initial state

\[ \Psi(x, 0) = \frac{1}{\sqrt{2\pi \hbar^2}} \exp\left(-\frac{i\pi p_0 (x-x_0)}{\hbar}\right) \left(\frac{x-x_0}{\hbar}\right)^2. \]

where our units are defined such that \( \hbar = 1 \), \( a = 2 \), \( x_0 = -\alpha/2 \), and \( \alpha = 1/4 \).

The probability density for the initial wave packet is shown below. If we increase \( \gamma \), the wave packet will still have the same initial probability density, but the wave packet will be built from higher energy states. We can determine the eigenstate, or value of \( n \), at which the wave packet is peaked by simply squaring the value of \( p_0 \).

To maximize the relative difference between even and odd revival times we chose \( V_0 = 25 \) and looked at wave packets peaked at energies corresponding to \( \gamma = 3, 25, \) and 50. This paper shows the results for a wave packet peaked at the \( n = 3 \) state, but similar behavior was found for the \( n = 25 \) and \( n = 50 \) cases.

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So second order perturbation theory predicts a different revival time for even-numbered eigenstates compared to odd-numbered eigenstates. This means that a wave packet composed of only even-numbered states will have a different revival time than a wave packet composed of only odd-numbered states. Values mean that a wave packet composed of both even-numbered and odd-numbered states may experience poor revivals because the even-numbered eigenstates are out of phase with the odd-numbered eigenstates. This is when the "odd" part of the wave packet is ready to revive, the "odd" part isn’t and vice versa. For specific parameter values we find clear evidence for this effect.

We find clear evidence for different revival times within a certain parameter regime. We also find that this difference in even and odd revival times is a leading cause in the poor revivals of the AISW.

Perturbation theory can be used to find the energy corrections for the AISW. First-order time independent perturbation theory gives the following formula for the energies of the AISW:

\[ E_n \approx \frac{\pi^2 \hbar^2}{2m(2a)^2} + \frac{V_0}{2} + \frac{\gamma n^2}{8\pi^2 \hbar^2}, \]

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