# Decay of wave packet revivals in the asymmetric infinite square well

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(Received 10 October 2010; accepted 28 December 2010)

In the infinite square well, any wave function will return to its initial state at integer multiples of the revival time. Most quantum systems do not exhibit perfect revivals, but some exhibit partial revivals in which the wave function returns close to its initial state. Subsequent partial revivals usually deteriorate in quality. We discuss the reasons for the perfect revivals in the infinite square well and how a small change in the potential disrupts the revivals. As an example, we examine partial revivals of a Gaussian wave packet in an infinite square well with a step. First-order and second-order perturbation theory show that the rate at which revivals decay depends on the location of the step. 
[DOI: 10.1119/1.3549714]

## I. INTRODUCTION

A wave packet is a quantum state that is, at least initially, well-localized in space. In a simple one-dimensional quantum well, a localized wave packet will mimic the motion of a classical particle over short time scales, oscillating with the same period as a classical particle whose energy is equal to the expectation value of the wave packet's energy. However, in most systems, the wave packet will also spread out over time. Eventually quantum interference effects produce a wave function whose evolution has no obvious relation to the motion of the classical particle. However, on longer time scales, the wave packet may return, at least approximately, to its initial state. The return of a wave packet to its well-localized initial state is known as a *revival* and occurs periodically with a period known as the *revival time*.

Wave packet revivals were first discussed in Ref. 1 and, in recent years, they have been the subject of much interest. Wave packet revivals have been observed experimentally,[2] but most studies of wave packet revivals involve simple models.[3] For example, coherent states in a simple harmonic oscillator system do not spread and have exact revivals after each classical oscillation period. In contrast, a wave packet in an infinite square well exhibits exact revivals at intervals that can be much longer than the classical period. In other one-dimensional quantum wells, wave packets exhibit only approximate revivals at the revival time.[4] Reference 5 provides a review of research in this area.

Much of the research on wave packet revivals has focused on simple square well systems.[6–8] In addition to the exact revivals we have described, a wave packet in an infinite square well can also display fractional revivals at shorter times.[9] In a finite square well, the wave packet revivals are not exact and the revival quality deteriorates with repeated revivals. However, the quality of these revivals improves near the *super-revival time*.[10,11]

Because the infinite square well has perfect revivals and the finite square well and most other systems do not, we ask how a small perturbation of the infinite square well alters the revival properties of wave packets. If the perturbation is a delta function at the center of the infinite square well, then it is possible for wave packets in this system to have perfect revivals, but only under certain conditions.[12] More generally, we would expect the perturbation to disrupt the revivals.

The goal of this paper is to examine the revival properties of wave packets in an infinite square well perturbed by a step function potential, a system sometimes referred to as the asymmetric infinite square well. The eigenstates of the asymmetric infinite square well can exhibit probability distributions that contradict classical intuitions[13] and some can have zero curvature.[14] The asymmetric infinite square well also has connections to problems in quantum chaos and periodic-orbit theory.[15–19] For our purposes, the asymmetric infinite square well serves as one of the simplest possible perturbations of the infinite square well and thus an ideal system in which to study how perturbations affect wave packet revivals.

In Sec. II, we will review some of the properties of wave packet revivals in the infinite square well system, emphasizing how the revival behavior depends on the structure of the energy spectrum. In Sec. III, we will use perturbation theory to determine how the energy spectrum of the infinite square well is altered by a step function perturbation and discuss how these alterations affect wave packet revivals. We discuss in Sec. IV numerical results on wave packet revivals in several versions of the asymmetric infinite square well and relate the numerical results to the predictions based on perturbation theory. In Sec. V, we will focus on the asymmetric infinite square well with the step at the center of the well. Section VI summarizes and discusses our results. The Appendix contains a series of problems to provide readers with the opportunity to work through some of the results presented in the paper. These problems are referenced in the relevant parts of the main text.

## II. REVIVALS IN THE INFINITE SQUARE WELL

The potential of the infinite square well can be written as

$$V(x)=\begin{cases}\infty, & x<-a\\ 0, & -a<x<b\\ \infty, & x>b.\end{cases} \tag{1}$$

The potential is written in this way in anticipation of the introduction of a step at $x=0$. The energy eigenvalues of the infinite square well are

$$E_n^{(0)}=\frac{\pi^2\hbar^2 n^2}{2m(a+b)^2}, \tag{2}$$

and the eigenstates are

$$\psi_n^{(0)}(x)=\sqrt{\frac{2}{a+b}}\,\sin\!\left(\frac{n\pi(x+a)}{a+b}\right). \tag{3}$$

  

The formal solution to the Schrödinger equation is

$$\Psi(x,t)=\sum_n c_n\psi_n(x)e^{-iE_nt/\hbar}, \tag{4}$$

where

$$c_n=\int_{-a}^{b}\Psi(x,0)\psi_n^*(x)dx \tag{5}$$

and $\Psi(x,0)$ is the initial wave function. For the infinite square well, $\psi_n(t)=\psi_n^{(0)}(t)$ and $E_n=E_n^{(0)}$, as given in Eqs. (3) and (2).

Equation (4) shows that the important quantities for determining the revival behavior of a wave function are the phases given by

$$\phi_n(t)=E_nt/\hbar. \tag{6}$$

If the phases $\phi_n$ have the same value at time $T_r$, for all $n$ such that $c_n\neq 0$, then the wave function $\Psi(x,T_r)$ is identical to the initial wave function $\Psi(x,0)$, except for a constant phase factor, which has no effect on the probability density.[20] Hence, the wave function experiences a revival at $t=kT_r$ for all integer values of $k$.

For the infinite square well,

$$T_r=\frac{4m(a+b)^2}{\pi\hbar} \tag{7}$$

and

$$\phi_n(T_r)=2\pi n^2=0 \bmod 2\pi \tag{8}$$

for all $n$ and therefore any wave function will have a revival at $t=T_r$ (problem 1). Note that $y=z \bmod 2\pi$ means that $y=z\pm 2\pi j$, where $j$ is an integer and, therefore, $y$ and $z$ are equivalent phases.

Although any wave function in the infinite square well will experience periodic revivals with the period in Eq. (7), some wave functions can have revivals at shorter intervals. For example, a wave function with odd spatial symmetry about $x=(b-a)/2$, which is composed only of even-numbered energy eigenstates, will experience revivals at $t=kT_r/4$ for integer $k$ (problem 2). Similarly, a wave function with even spatial symmetry, composed only of odd-numbered eigenstates, will experience revivals at $t=kT_r/8$ for integer $k$ (problem 3).

What happens at these smaller time intervals if the wave function is composed of both even-numbered and odd-numbered eigenstates? At $t=T_r/8$, we would not expect a revival because the even-numbered eigenstates are not in phase with each other. But what about $t=T_r/4$? At this time, the even-numbered eigenstates are all in phase with each other and the odd-numbered eigenstates are all in phase with each other. The even-numbered states have phase 0 and the odd-numbered states have phase $\pi/2$ (see problems 2 and 3). Similarly, at $t=T_r/2$, the even-numbered states have phase 0 and the odd-numbered states have phase $\pi$.

The difference in phases at $t=T_r/2$ leads to an interesting result. We can write the solution in Eq. (4) as a sum over the even-numbered eigenstates plus a sum over the odd-numbered eigenstates

$$\Psi(x,t)=\Psi^{\text{even}}(x,t)+\Psi^{\text{odd}}(x,t). \tag{9}$$

Note that the function $\Psi^{\text{even}}$ has odd symmetry and $\Psi^{\text{odd}}$ has even symmetry. At $t=T_r/2$, all the even-numbered eigenstates in Eq. (4) are multiplied by a factor of $e^0=1$ and all the odd-numbered states are multiplied by $e^{\pi}=-1$. As a result, the full wave function at $t=T_r/2$ is a mirror image of the initial state $\Psi(x,0)$, with an overall phase factor of $-1$ (problem 4). This phenomenon is known as a *mirror revival*. Similarly, the wave function at $t=T_r/4$ is a superposition of the original state and its mirror image, an example of what is known as a *fractional revival*.[9]

The revival behavior of a wave packet in the infinite square well can be illustrated numerically. We let $b=a$ (so that the well is centered at $x=0$) and choose the initial state to be a Gaussian wave packet

$$\Psi(x,0)=\left(\frac{1}{\rho^2\hbar^2\pi}\right)^{1/4}\exp\left[-\frac{ip_0(x-x_0)}{\hbar}-\frac{(x-x_0)^2}{2\rho^2\hbar^2}\right], \tag{10}$$

where $x_0$ and $p_0$ specify the mean position and momentum of the packet and $\rho$ sets the width of the packet. We choose $x_0=-a/2$ (so that the wave packet starts on the left side of the well) and $p_0=\sqrt{2mE_{15}^{(0)}}$ (so that the greatest contribution to the wave packet comes from the $n=15$ eigenstate). The solution to the Schrödinger equation for this initial condition can be generated using Eqs. (4) and (5) using a reduced Hilbert space approach, in which the sum in Eq. (4) becomes finite because $c_n\approx 0$ for all but a finite number of values of $n$.[21] Figure 1 shows the probability density for this initial condition at times $t=T_r/4$, $T_r/2$, and $T_r$, illustrating a fractional revival, a mirror revival, and a full revival.

Another way to examine the revival behavior for this wave packet is to calculate the autocorrelation function

$$A(t)=\int_{-a}^{b}\Psi^*(x,t)\Psi(x,0)dx. \tag{11}$$

[See problem 5 for an alternate way of expressing $A(t)$.] This quantity measures the overlap of the initial wave packet with the wave packet at time $t$. A perfect revival at time $t$ would result in $|A(t)|^2=1$. Figure 2(a) shows a plot of the numerically computed values of $|A(t)|^2$ versus time for the initial state given in Eq. (10). The sharp peak at $t=T_r$ corresponds to the first full revival of the wave packet.

To illustrate the revivals of wave packets composed of only even-numbered or odd-numbered eigenstates, we must use a different initial condition. Instead of the Gaussian in Eq. (10), we project this wave packet onto the set of even-numbered (or odd-numbered) eigenstates and then normalize the resulting wave function. Figure 2(b) shows the autocorrelation function for the wave packet projected onto odd-numbered states and clearly shows revival peaks at intervals of $T_r/8$. Likewise, Fig. 2(c) shows the results for a projection onto even-numbered states, illustrating the revivals at intervals of $T_r/4$.

## III. PERTURBING THE INFINITE SQUARE WELL

To investigate how a perturbation of the infinite square well affects these revivals, we examine the asymmetric infi-

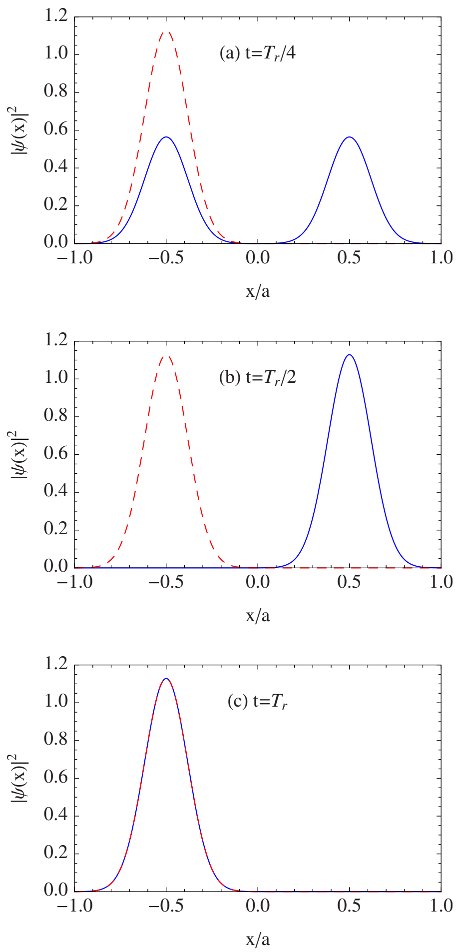

Fig. 1. Gaussian wave packet in the infinite square well. Shown is the probability density at (a) $t=T_r/4$ (fractional revival), (b) $t=T_r/2$ (mirror revival), and (c) $t=T_r$ (full revival). The initial state is shown as a dashed curve.

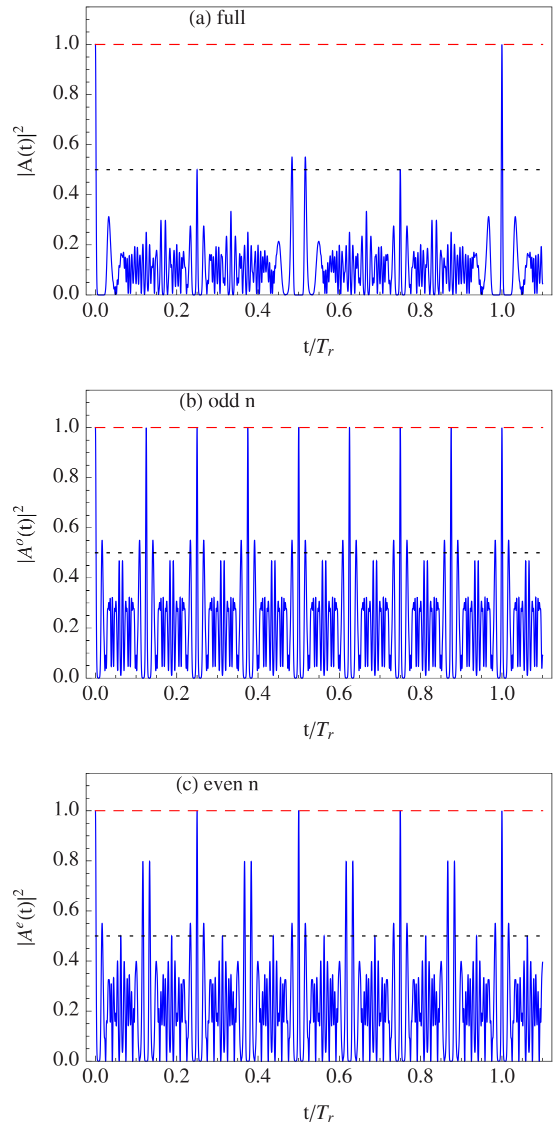

Fig. 2. Revivals in the infinite square well. The autocorrelation functions are shown for (a) a Gaussian wave packet, (b) a wave packet composed of odd-numbered (even symmetry) states, and (c) a wave packet composed of even-numbered (odd symmetry) states.

nite square well. The latter is an infinite square well with a discontinuous potential step. The potential energy for the asymmetric infinite square well is

$$V(x)=\begin{cases}\infty, & x\le -a\\ 0, & -a<x<0\\ V_0, & 0<x<b\\ \infty, & x\ge b.\end{cases} \tag{12}$$

In Sec. II, we found that the revival behavior of a wave packet depends on the structure of the energy spectrum for the eigenstates that compose the initial wave function. To understand how revivals in the asymmetric infinite square well differ from those in the infinite square well, we need to determine how the energy spectrum of the asymmetric infinite square well differs from Eq. (2). For this purpose, we turn to perturbation theory.

Standard (Rayleigh–Schrödinger) time-independent perturbation theory proceeds by writing the Hamiltonian for the system as $H=H_0+V_p$, where $H_0$ is the Hamiltonian of the "unperturbed" system and $V_p$ is the perturbation. For the asymmetric infinite square well, the unperturbed system is an infinite square well with the potential given in Eq. (1). The perturbation is

$$V_p(x)=\begin{cases}0, & x<0\\ V_0, & 0<x<b\\ 0, & x>b.\end{cases} \tag{13}$$

By using first-order perturbation theory (problem 6), we find that the energy eigenvalues of the asymmetric infinite square well are

$$E_n \approx \frac{\pi^2\hbar^2 n^2}{2m(a+b)^2}+\frac{bV_0}{a+b}+\frac{V_0}{2n\pi}\sin\left(\frac{2n\pi a}{a+b}\right). \tag{14}$$

How does this alteration of the energy spectrum affect the revival behavior of a wave packet? One way to answer this question is to consider the *phase lines* for the system. A phase line is the remainder of the function $\phi_n(t)$ [see Eq. (6)] when it is divided by $2\pi$. For a wave packet to have a revival, the phase lines for those values of $n$ that contribute to the initial state must all intersect at some time $t$.

Consider two phase lines with quantum numbers $n$ and $k$. These phase lines intersect when

$$E_n t = E_k t + 2\pi\hbar j, \tag{15}$$

where $j$ is any integer. For the infinite square well, the phase lines intersect at $t=T_r$

$$E_n^{(0)} T_r = E_k^{(0)} T_r + 2\pi\hbar j. \tag{16}$$

The perturbation changes the energies and thus alters the slopes of the phase lines and shifts the intersection time to $T_r + \delta T_r$.

The new condition for the intersection is

$$(E_n^{(0)} + \delta E_n)(T_r + \delta T_r) = (E_k^{(0)} + \delta E_k)(T_r + \delta T_r) + 2\pi j, \tag{17}$$

where $\delta E$ is the change in the energy eigenvalue produced by the perturbation. If the change is sufficiently small, then the value of $j$ in Eqs. (15) and (16) will be the same. If we use the first-order perturbation theory result for $\delta E$, we find (problem 7)

$$\delta T_r = -\frac{V_0 T_r^2}{4\pi^2\hbar(n^2-k^2)}\left[\frac{1}{n}\sin\left(\frac{2\pi an}{a+b}\right) - \frac{1}{k}\sin\left(\frac{2\pi ak}{a+b}\right)\right]. \tag{18}$$

For $b=a$, first-order perturbation theory predicts that all phase lines intersect each other at $t=T_r$ and the wave packet still has an exact revival at this time (problem 8). That is, the first-order correction has no effect on the revival behavior for $b=a$. If $b\neq a$, the situation is different. The value of $\delta T_r$ given by Eq. (18) can be positive or negative, depending on the values of $n$ and $k$. Some of the phase intersections will be shifted to later times and some to earlier times and there will no longer be a single time at which all of the phase lines intersect. Instead, the intersections are spread over an interval around $t=T_r$. From Eq. (18) we see that the size of this interval is proportional to $V_0$.

For an initial state such as a Gaussian wave packet, only a small number of energy eigenstates contribute to the wave function. To determine the revival behavior of the wave packet, we need only to be concerned with intersections among the phase lines for the contributing eigenstates. Therefore, the intersections that are relevant for revival behavior are between phase lines with values of $n$ and $k$ that are close to each other. For these conditions, Eq. (18) indicates that the spread of intersection times around $t=T_r$ is smaller when $n$ is large (and therefore $k$ is large).

For sufficiently small perturbations, the intersections remain close to $t=T_r$. Thus, at $t=T_r$, the phases of the states contributing to a wave packet are close to each other, though not quite equal, and the wave packet experiences an approximate revival. At subsequent revival times, the intersections become increasingly spread out and the quality of the revival (as measured by the value of $|A(t)|^2$ at the revival time) will decay. This decay proceeds more rapidly for large values of $V_0$ and wave packets composed of low-energy eigenstates. The decay proceeds slowly for small $V_0$ and high-energy wave packets, approximating the behavior of the infinite square well as expected.

## IV. REVIVALS IN THE ASYMMETRIC INFINITE SQUARE WELL

To test the predictions arising from Eq. (18), we numerically examine the revivals of a Gaussian wave packet in the asymmetric infinite square well. The energy eigenvalue problem can be solved analytically (problem 9).[13] The equation for the energy eigenvalues is

$$Q\cos(Qa)\sin(qb) + q\cos(qb)\sin(Qa) = 0, \tag{19}$$

where $Q \equiv \sqrt{2mE}/\hbar$ and $q \equiv \sqrt{2m(E-V_0)}/\hbar$.

Equation (19) is a transcendental equation with an infinite number of solutions. However, we can ignore all eigenvalues $E_n$ for which $c_n \approx 0$ [with $c_n$ defined as in Eq. (5)]. For the Gaussian initial condition, this approximation leaves only a finite number of eigenvalues to compute. An efficient procedure for computing solutions to Eq. (19) is described in Refs. 15 and 19. We compute energy eigenvalues for the system with different values of $a$ and $b$, with the value of $a+b$ held fixed so that the unperturbed energies (and revival times) are the same in all cases. With these eigenvalues and the eigenstates given in Eq. (A10), we use Eqs. (4) and (5) to solve the Schrödinger equation for the same initial state examined in Sec. II (a Gaussian wave packet peaked at energy $E_{15}$).

We first examine the phase lines for various perturbations. Figure 3 shows phase lines with $n=10,11,\ldots,20$ (the eigenstates that contribute most strongly to the wave function) at times close to $t=T_r$. Figure 3(a) shows the phase lines for $b=a/\sqrt{2}$, with $V_0=0.25$ (in units scaled so that $a+b=6$, $\hbar=1$, and $m=0.5$). Here we see the jumble of intersections predicted by Eq. (18). Some intersections occur at $t<T_r$ and others occur at $t>T_r$. In this typical case, we expect an approximate revival at $t=T_r$, because the phases are all close to each other at that time, and the quality of the revival to decay during subsequent revivals.

In contrast, we know that for $b=a$, all the phase lines should intersect at $t=T_r$ according to first-order perturbation theory. We will examine the case $b=a$ in Sec. V, but here we consider the case $b\approx a$. Figure 3(b) shows the phase lines for $b=0.95a$. The phase lines form two groups, one of which intersects at $t>T_r$ and the other at $t<T_r$. Overall, the phases are still spread apart at $t=T_r$ and we expect to see significant decay of revivals. Figure 3(c) shows the phase lines for $b=0.99a$. In this case, the lines also form two groups, but these groups are much closer together at $t=T_r$. We expect the decay of the wave packet to be slower for this case than for $b=0.95a$ or $b=a/\sqrt{2}$.

Figure 3 provides some predictions for the revival behavior for these cases. To verify these predictions, we examine the autocorrelation function as a function of time. Figure 4 shows a plot of $|A(t)|^2$ for $b=a/\sqrt{2}$. There are well-defined peaks at multiples of $T_r$ and also a noticeable decay. The peak at $t=T_r$ seems to reach $|A(t)|^2=1$ and the peak at $t=4T_r$ falls noticeably short of unity.

To study the decay of revivals over longer time periods, we plot $|A(kT_r)|^2$ for integer values of $k\leq 20$. Figure 5 shows the results for several perturbations. The case $b=a$ shows little or no decay as expected from first-order perturbation theory. A typical case with $b=a/\sqrt{2}$ shows significant decay over 20 revivals. The case $b=0.95a$ shows a decay that is even more rapid. The formation of the phase lines into groups for $b=0.95a$ leads to large phase differences between even-numbered and odd-numbered states, which contributes

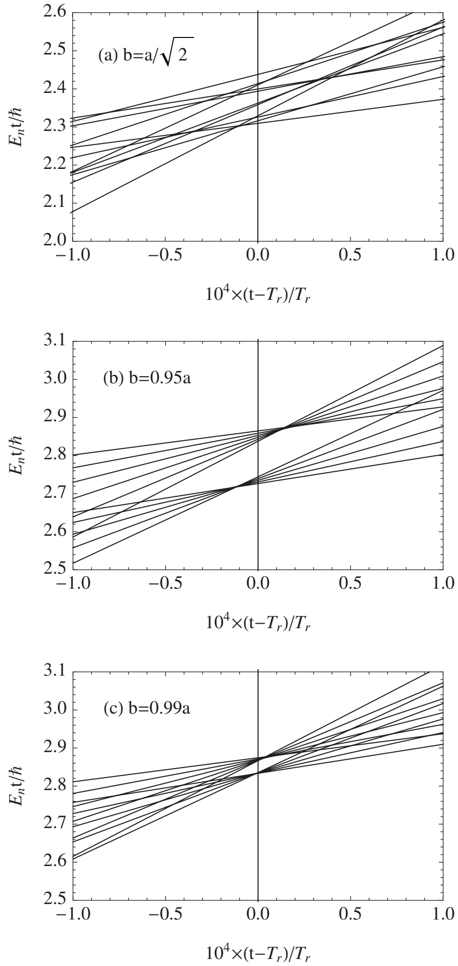


Fig. 3. Phase lines for eigenstates near $n_0=15$ in the asymmetric infinite square well. Shown are phase lines near the infinite square well revival time for $V_0=0.25$ (in scaled units) with (a) $b=a/\sqrt{2}$, (b) $b=0.95a$, and (c) $b=0.99a$.

to a rapid decay of the revivals. In contrast, the case $b=0.99a$ shows a slower decay than the typical case. Here the two groups of phase lines have become close together and approach a common intersection point, approximating the seemingly decay-free behavior of the $b=a$ case.

In these cases, $V_0=0.25$ in the scaled units defined previously. Figure 5 also shows results for $b=a/\sqrt{2}$ with $V_0=0.5$ in our scaled units. For $V_0=0.5$, we see that the decay is

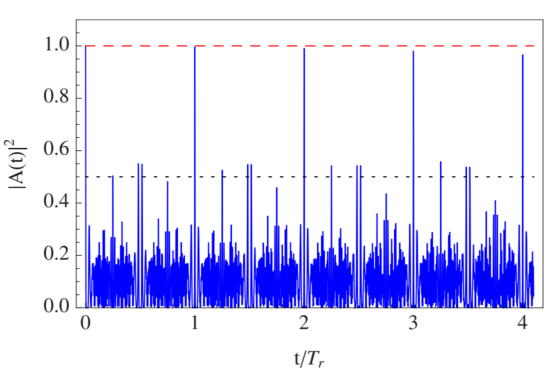


Fig. 4. Autocorrelation function for a Gaussian wave packet for the asymmetric infinite square well with $b=a/\sqrt{2}$ and $V_0=0.25$ (in scaled units).

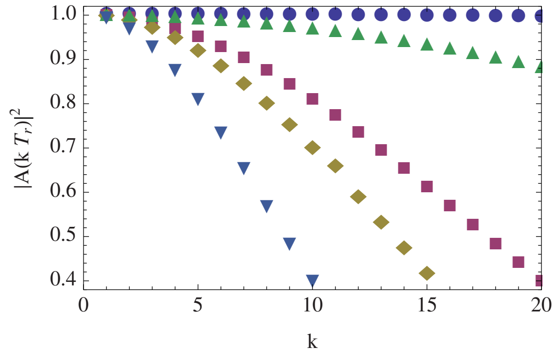


Fig. 5. Decay of revivals for the asymmetric infinite square well. The absolute square of the autocorrelation function is shown at $t=kT_r$ for $k=1,2,\ldots,20$ with $V_0=0.25$ and $b=a/\sqrt{2}$ (□), $b=a$ (○), $b=0.95a$ (◇), and $b=0.99a$ (△), and with $V_0=0.5$ and $b=a/\sqrt{2}$ (▽). Values of $V_0$ are given in scaled units.

twice as fast as for $V_0=0.25$ with the same value of $b$. This behavior is consistent with our perturbation theory prediction that the spread of the phases, and thus the decay rate, is proportional to $V_0$.

## V. THE CASE $b=a$

At first glance, Fig. 5 indicates that there is no decay of revivals for $b=a$. A closer look reveals that the revivals decay very slowly compared to the other cases with $b\neq a$. The reasons for this decay are revealed by a closer look at the phase lines for $b=a$ shown in Fig. 6. The lines do not all intersect at $t=T_r$, as predicted by first-order perturbation theory.

To understand the results in Fig. 6, we examine the second-order correction to the energies (problem 10). For large values of $n$, the second-order energies can be approximated (problem 11) as[18]

$$E_n \approx \frac{\pi^2\hbar^2 n^2}{8ma^2} + \frac{V_0}{2} + \frac{\gamma_n m a^2 V_0^2}{2\pi^2\hbar^2 n^2}, \tag{20}$$

where

$$\gamma_n = \begin{cases} 3, & \text{even } n \\ -1, & \text{odd } n. \end{cases} \tag{21}$$

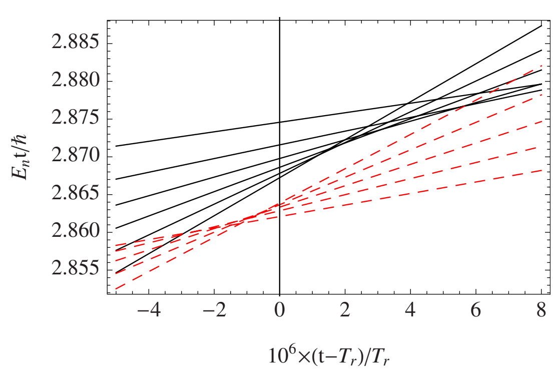


Fig. 6. Close-up of phase lines for $b=a$. Lines for even-numbered states are solid and lines for odd-numbered states are dashed.

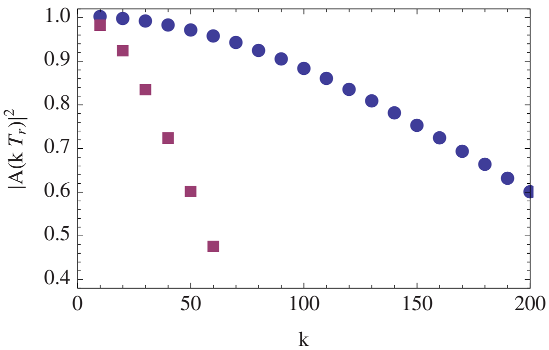


Fig. 7. Decay of revivals for $b=a$. The absolute square of the autocorrelation function is shown at $t=kT_r$ for $k=10,20,\ldots,200$ with $V_0=0.25$ (○) and $V_0=0.5$ (□). Values of $V_0$ are given in scaled units.

By using Eq. (20), we can show (problem 12) that for $b=a$, the intersection of the phase curves for eigenstates $n$ and $k$ is shifted by

$$\delta T_r = \frac{ma^2V_0^2T_r^2}{4\pi^3\hbar^3n^2k^2} \times \begin{cases} -1, & k \text{ and } n \text{ odd} \\ 3, & k \text{ and } n \text{ even} \\ \dfrac{k^2+3n^2}{n^2-k^2}, & k \text{ even, } n \text{ odd} \\ \dfrac{3k^2+n^2}{n^2-k^2}, & k \text{ odd, } n \text{ even.} \end{cases} \tag{22}$$

Equation (22) shows that intersections among odd-numbered states are shifted to $t<T_r$ and intersections among even-numbered states are shifted (by a larger amount) to $t>T_r$. This behavior is consistent with the results shown in Fig. 6. The phase lines form into two groups, much as we saw for $b=0.95a$ and $b=0.99a$. However, the difference in the intersection time between the two groups is much smaller for $b=a$. Hence, for $b=a$, we expect revivals to decay over time, but at a much slower rate than for $b\neq a$. Equation (22) shows that the spread of intersection times for $b=a$ is proportional to $V_0^2$, whereas it was proportional to $V_0$ for $b\neq a$.

These predictions from second-order perturbation theory are confirmed by computations of the revival behavior for a Gaussian wave packet for $b=a$. We followed the same procedure given in Sec. IV and computed $|A(kT_r)|^2$ for several values of $k$. Figure 7 shows the results for $V_0=0.25$ and $V_0=0.5$ (in scaled units). These results illustrate the decay of the revivals and that the decay rate is proportional to $V_0^2$.

Equation (22) shows that the perturbation divides the energy eigenvalues into two groups, according to whether the state number is even or odd. Figure 6 shows that the odd-numbered phase lines nearly intersect at a common point just before $t=T_r$. Similarly, the even-numbered phase lines nearly intersect at a common point just after $t=T_r$, although the intersections are not as close to each other as for the odd-numbered states. Thus, a subpacket composed only of odd-numbered eigenstates exhibits revivals with a very slow decay rate. A subpacket composed of only even-numbered states also decays slowly, though faster than an odd-numbered wave packet. A packet composed of both even and odd states decays more rapidly because the even phases become separated from the odd phases over time.

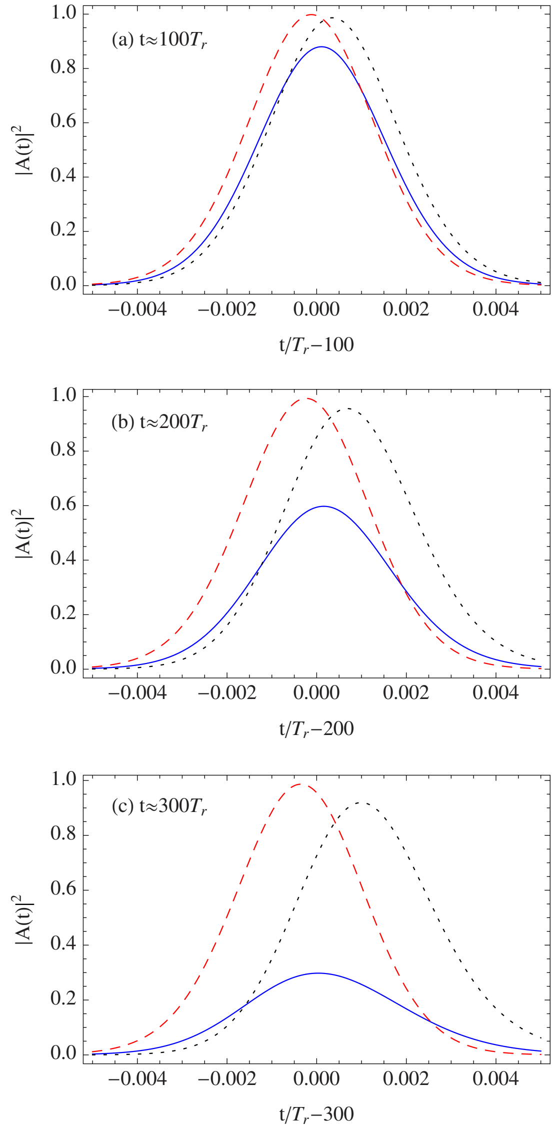


Fig. 8. Autocorrelation function for a full wave packet (solid), a subpacket of odd-numbered states (dashed), and a subpacket of even-numbered states (dotted) for $b=a$. $|A(t)|^2$ for all three packets are shown for (a) $t\approx 100T_r$, (b) $t\approx 200T_r$, and (c) $t\approx 300T_r$.

We investigated this behavior by computing the autocorrelation function for a Gaussian wave packet (as described previously) and for subpackets composed of only odd-numbered or only even-numbered eigenstates. The subpackets are constructed by projecting the Gaussian wave packet onto the even-numbered (or odd-numbered) eigenstates and then normalizing the result. Figure 8 shows a close-up of $|A(t)|^2$ for all three wave functions near the revival times ($t=100T_r$, $200T_r$, and $300T_r$). It is clear from Fig. 8 that the even subpacket decays faster than the odd subpacket, as expected. The full wave packet decays much faster than either of the subpackets. The autocorrelation function for the full wave packet is bracketed by those for the subpackets, showing that the faster decay of the full packet is caused by the even and odd eigenstates becoming progressively out of phase with each other.

A closer examination of the revivals of the even-subpackets and odd-subpackets reveals that the time between revivals as determined by the maxima of $|A(t)|^2$ does not

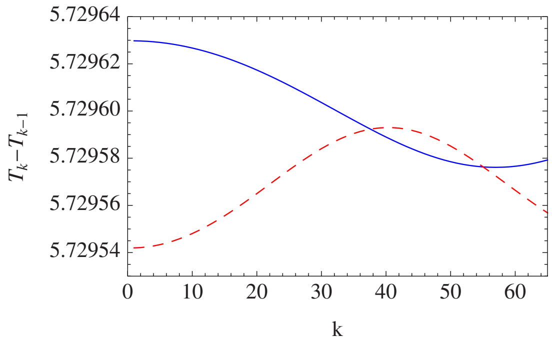


Fig. 9. Revival times for subpackets of even-numbered (solid) and odd-numbered (dashed) states. The plot shows the difference between the time for the $k$th and $(k+1)$st revival peaks as a function of $k$ for both subpackets.

remain fixed. Instead, the time between subsequent revivals oscillates sinusoidally as shown in Fig. 9. Although these oscillations have a very small amplitude, they are sufficiently large for the time between revivals of an odd subpacket to occasionally be greater than the time between revivals for an even subpacket.

## VI. DISCUSSION

Perfect revivals occur when the phase lines associated with the energy eigenvalues all intersect at a common point. Perturbing the infinite square well by adding a potential step within the well disrupts these perfect revivals because it shifts the intersections of the phase lines so that they no longer occur at a common point. A wave packet may still undergo revivals in the perturbed system if the perturbation is small compared to the energies of the eigenstates that contribute to the wave packet. In this case, the intersections of the phase lines are not spread far apart. These revivals are approximate and decay over time as the intersection points become increasingly spread apart with each subsequent revival.

We examined only the basic behavior of wave packet revivals in this perturbed infinite square well. The oscillations of the revival times shown in Fig. 9 are still unexplained. This system might exhibit super-revivals on much longer time scales than those we have studied. There may be interesting connections between the revivals in this system and periodic-orbit theory.[15,19] Perhaps wave packets with support on zero-curvature eigenstates exhibit new revival behaviors.[14] The rich behavior of this simple system has yet to be fully explored.

## ACKNOWLEDGMENTS

The authors would like to thank Mario Belloni and Amal K. Das for helpful discussions about the asymmetric infinite square well. They would also like to thank Robert Boston for his preliminary work on the topic of this paper.

## APPENDIX: PROBLEMS

(1) Show that Eq. (8) for the infinite square well holds for any value of $n$ if $T_r$ is defined as in Eq. (7).

(2) Show that the energy eigenvalues of the infinite square well for even-numbered eigenstates with $n=2i$ can be written as

$$E_i^{\text{even}} = \frac{2\pi^2\hbar^2 i^2}{m(a+b)^2}. \tag{A1}$$

Show that the phases of these even-numbered eigenstates at $t=T_r/4$ are given by

$$\phi^{\text{even}}(T_r/4) = 2\pi i^2 = 0 \bmod 2\pi \tag{A2}$$

and therefore a wave function composed only of even-numbered eigenstates experiences revivals at $t=kT_r/4$ for all integer values of $k$.

(3) Show that the energy eigenvalues of odd-numbered eigenstates in the infinite square well can be written as

$$E_i^{\text{odd}} = \frac{\pi^2\hbar^2(2i+1)^2}{2m(a+b)^2}. \tag{A3}$$

Show that the phase of these states at $t=T_r/8$ is

$$\phi^{\text{odd}}(T_r/8) = i(i+1)\pi + \pi/4 = \pi/4 \bmod 2\pi \tag{A4}$$

and therefore a wave function composed only of odd-numbered eigenstates experiences revivals at $t=kT_r/8$ for all integer values of $k$.

(4) Show that the wave function for the infinite square well at $t=T_r/2$ can be written as

$$\Psi(x,T_r/2) = \Psi^{\text{even}}(x,0) - \Psi^{\text{odd}}(x,0) = -\Psi(-x,0). \tag{A5}$$

(5) Show that the autocorrelation function defined in Eq. (11) can be written as

$$A(t) = \sum_n |c_n|^2 e^{iE_n t/\hbar}. \tag{A6}$$

This form is much more convenient for numerical computations.

(6) The first-order correction to the energy eigenvalues is given by

$$E_n^{(1)} = \langle \psi_n^{(0)} | V_p | \psi_n^{(0)} \rangle. \tag{A7}$$

Use Eqs. (A7), (3), and (13) to derive Eq. (14). Show that for $b=a$, Eq. (14) reduces to

$$E_n \approx \frac{\pi^2\hbar^2 n^2}{8ma^2} + \frac{V_0}{2}. \tag{A8}$$

(7) Solve Eq. (17), neglecting the small $\delta E\,\delta T$ terms, to show that

$$\delta T_r = -\frac{(\delta E_n - \delta E_k)T_r}{E_n^{(0)} - E_k^{(0)}}. \tag{A9}$$

Use the first-order perturbation theory results in Eq. (14) to derive Eq. (18).

(8) Show that if $b=a$, then $\delta T_r$ from Eq. (18) is zero for all $n$ and $k$.

(9) Show that for $E > V_0$ the energy eigenstates for the asymmetric infinite square well are of the form

$$\psi(x) = \begin{cases} A \sin[Q(x+a)] & -a < x \le 0 \\ B \sin[q(x-b)] & 0 < x < b. \end{cases} \tag{A10}$$

From the requirement that $\psi$ and $d\psi/dx$ be continuous at $x=0$, derive the energy eigenvalue equation given in Eq. (19).

(10) The second-order perturbation correction to the energy eigenvalues is given by

$$E_n^{(2)} = \sum_{k\neq n} \frac{|\langle \psi_k^{(0)}|V_p|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}}. \tag{A11}$$

For the asymmetric infinite square well with $b=a$, show that

$$\langle \psi_k^{(0)}|V_p|\psi_n^{(0)}\rangle = \frac{V_0}{\pi}\left(\frac{\sin[(k+n)\pi/2]}{k+n} - \frac{\sin[(k-n)\pi/2]}{k-n}\right). \tag{A12}$$

Show that the absolute square of this matrix element simplifies to

$$|\langle \psi_k^{(0)}|V_p|\psi_n^{(0)}\rangle|^2 = \begin{cases} \dfrac{4V_0^2k^2}{\pi^2(k^2-n^2)^2}, & \text{odd } n \text{ and even } k \\ \dfrac{4V_0^2n^2}{\pi^2(k^2-n^2)^2}, & \text{even } n \text{ and odd } k \\ 0, & \text{otherwise.} \end{cases} \tag{A13}$$

Use this result to show that the second-order correction to the asymmetric infinite square well energies for $b=a$ is

$$E_n^{(2)} = \begin{cases} \dfrac{8mV_0^2(a+b)^2n^2}{\pi^4\hbar^2}\displaystyle\sum_{\text{odd } k}\frac{1}{(n^2-k^2)^3}, & \text{even } n \\ \dfrac{8mV_0^2(a+b)^2}{\pi^4\hbar^2}\displaystyle\sum_{\text{even } k}\frac{k^2}{(n^2-k^2)^3}, & \text{odd } n. \end{cases} \tag{A14}$$

(11) Show that for large $n$, Eq. (A14) can be approximated by

$$E_n^{(2)} \approx \frac{\gamma_n m a^2 V_0^2}{2\pi^2\hbar^2 n^2}, \tag{A15}$$

with $\gamma_n$ defined in Eq. (21). (This problem is challenging. The solution can be found in Ref. 18.)

(12) Derive Eq. (22) from Eqs. (A9) and (20).

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