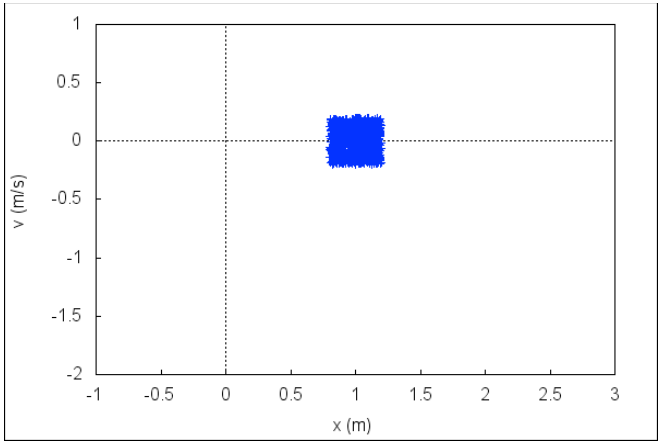
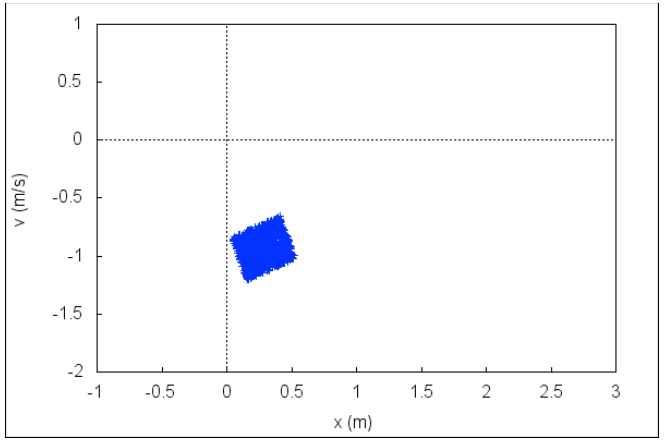
# Learning About Liouville's Theorem with ODE Solver Algorithms




Todd Timberlake

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Physics & Astronomy

# Computation with Physics

- Time is limited to teach computation.
- Teach computational topics that also illustrate physical principles.
- ODE solver algorithms can be used to illustrate Liouville's theorem in an accessible way (no Hamiltonian mechanics!).
- Liouville's theorem: in conservative systems the phase space volume of an ensemble of points in conserved

# Harmonic Oscillator no damping

Euler-Cromer algorithm

$$
\begin{aligned}
v_{n+1} &= v_n - \frac{k}{m} x_n \Delta t \\
x_{n+1} &= x_n + v_{n+1} \Delta t \\
&= x_n + v_n \Delta t - \frac{k}{m} x_n \Delta t^2
\end{aligned}
$$

# The Jacobian Matrix and Determinant

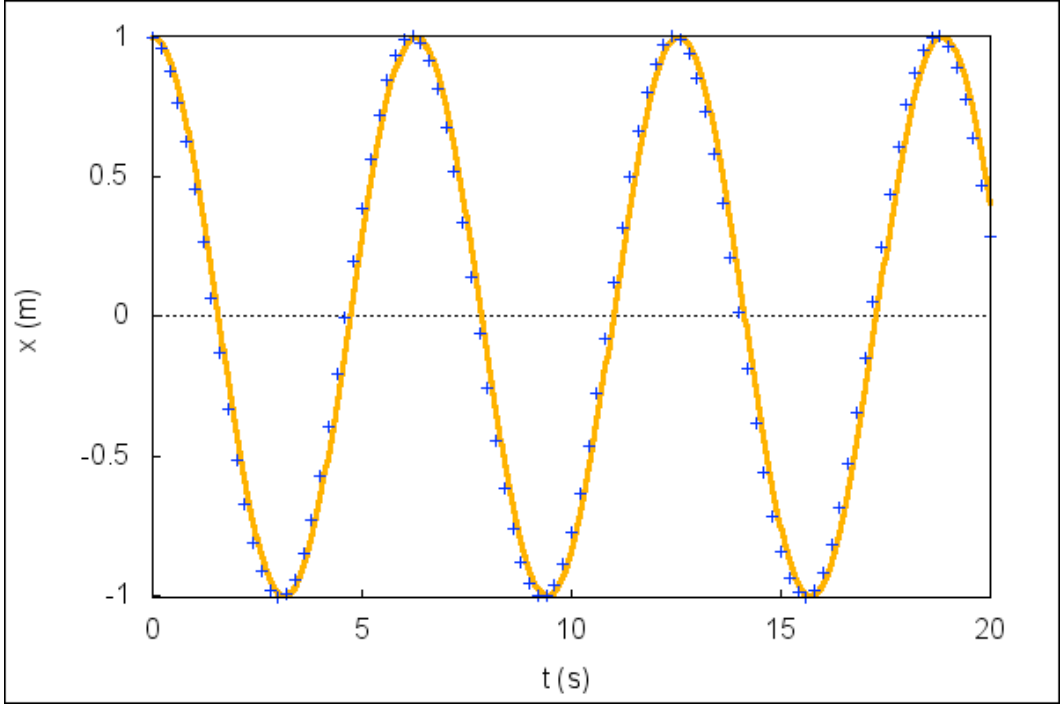
$$dx_{n+1}dv_{n+1} = |J|dx_n dv_n$$

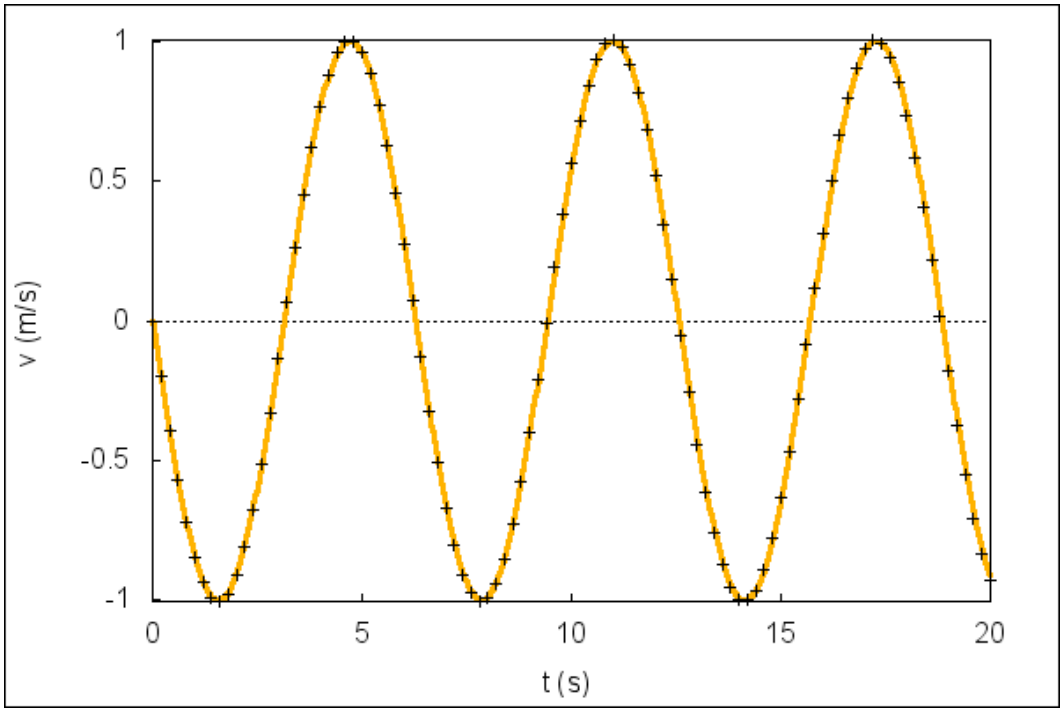
$$J = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial v_n} \\ \frac{\partial v_{n+1}}{\partial x_n} & \frac{\partial v_{n+1}}{\partial v_n} \end{pmatrix} = \begin{pmatrix} 1 - k\Delta t^2/m & \Delta t \\ -k\Delta t/m & 1 \end{pmatrix}$$

$$|J| = 1$$

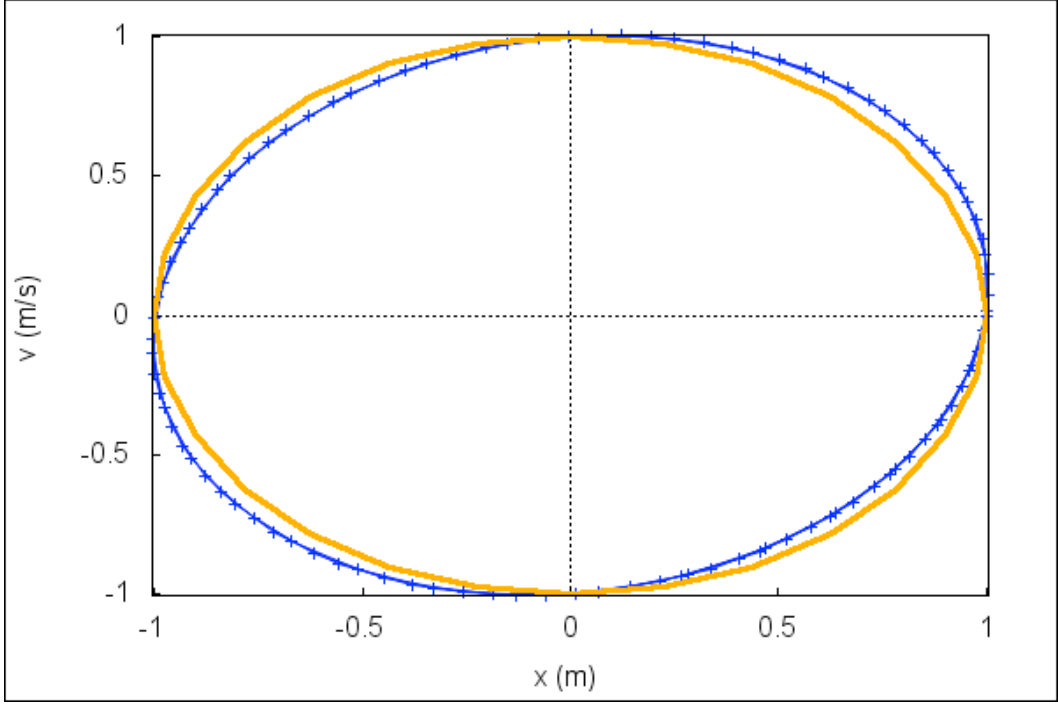
# Position and Velocity Plots

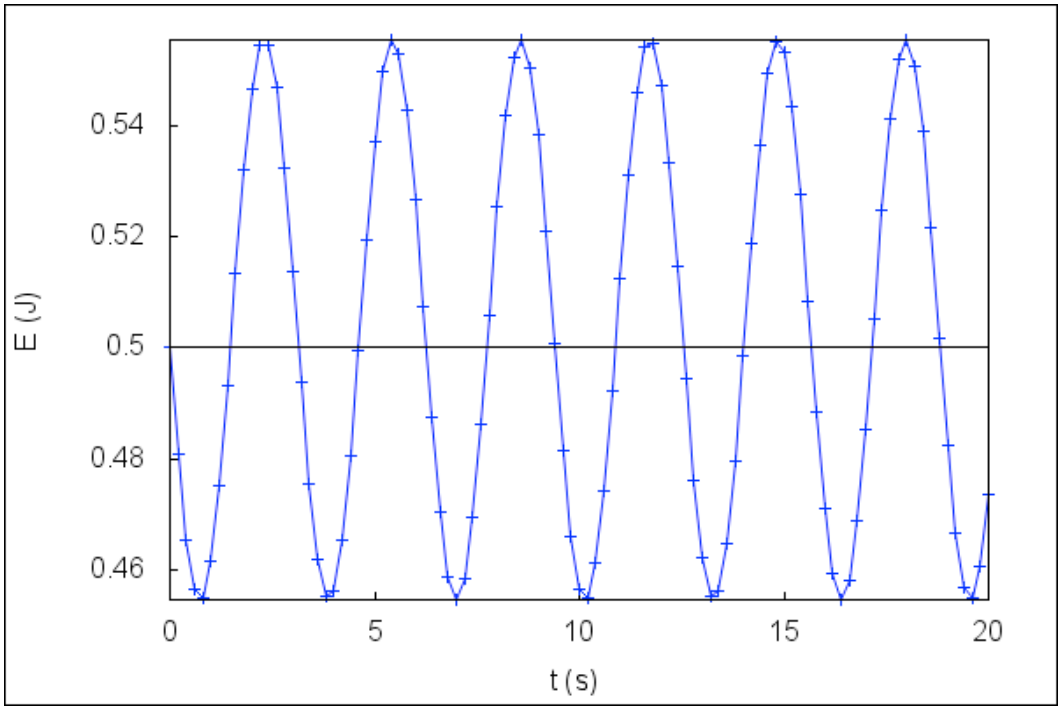
Note: All numerical results and plots were generated with the Maxima computer algebra system. Link to code will be given at the end.



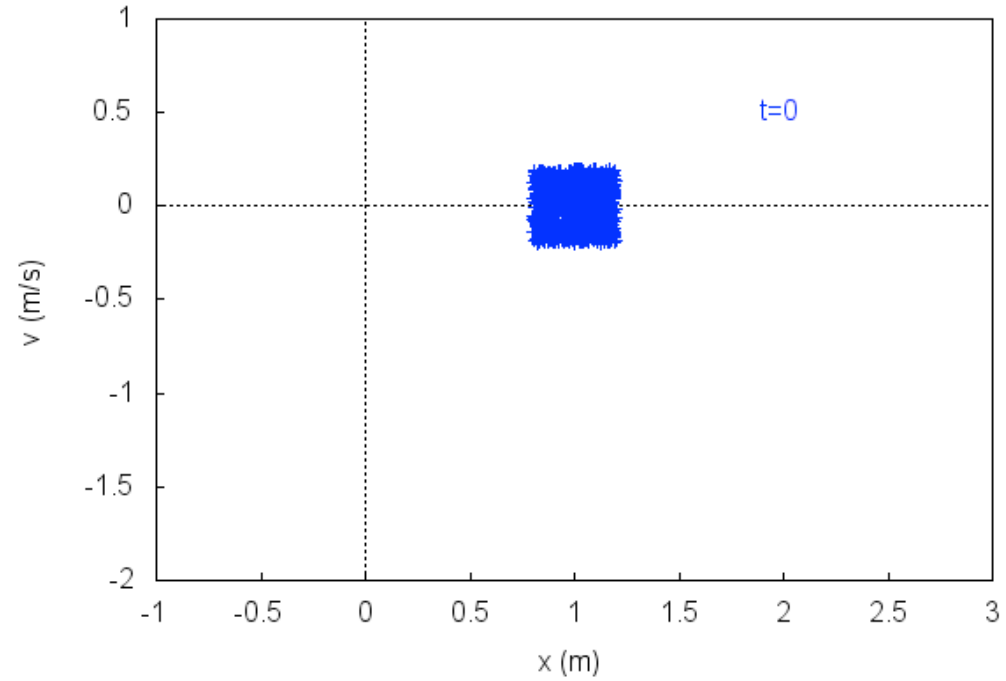



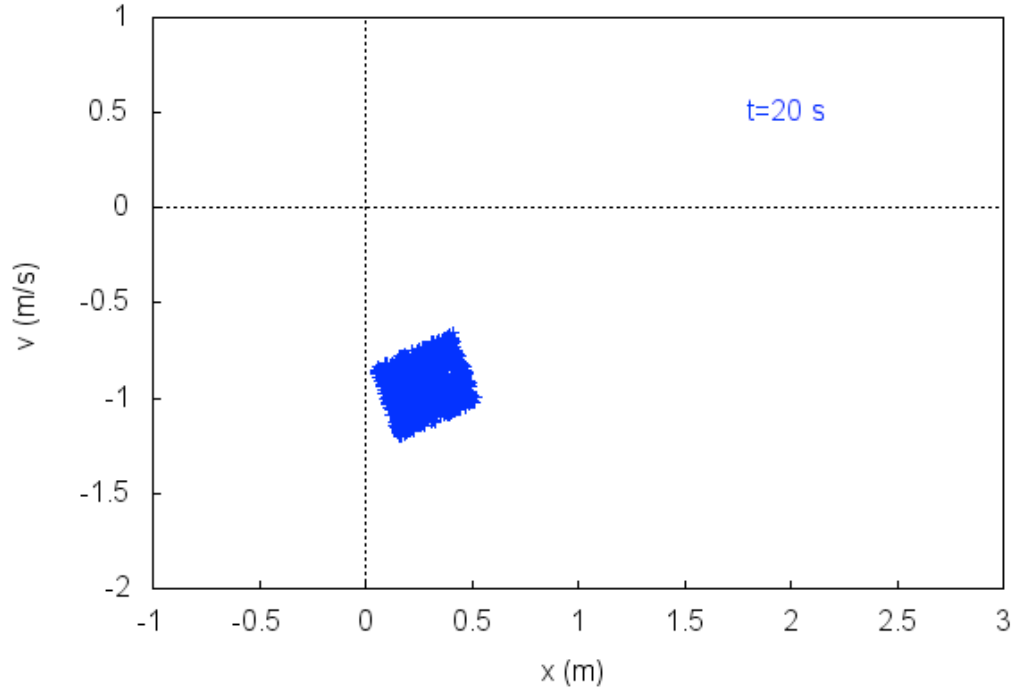
# Phase Space Path and Energy Plot



# Evolution of Ensemble in Phase Space



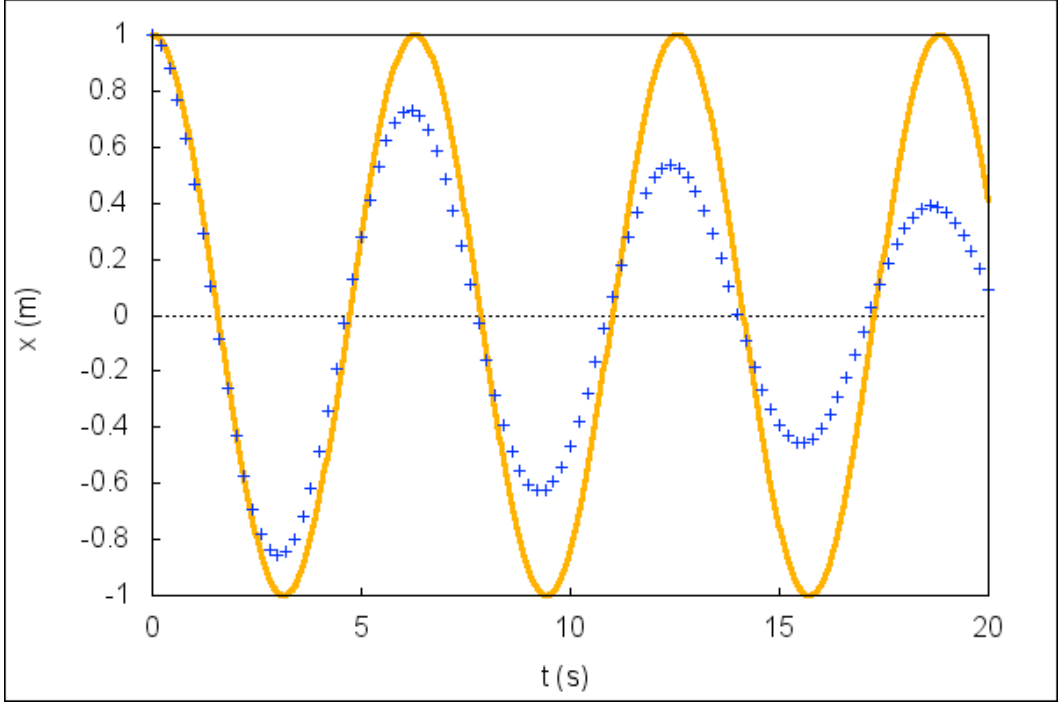
# Harmonic Oscillator linear damping

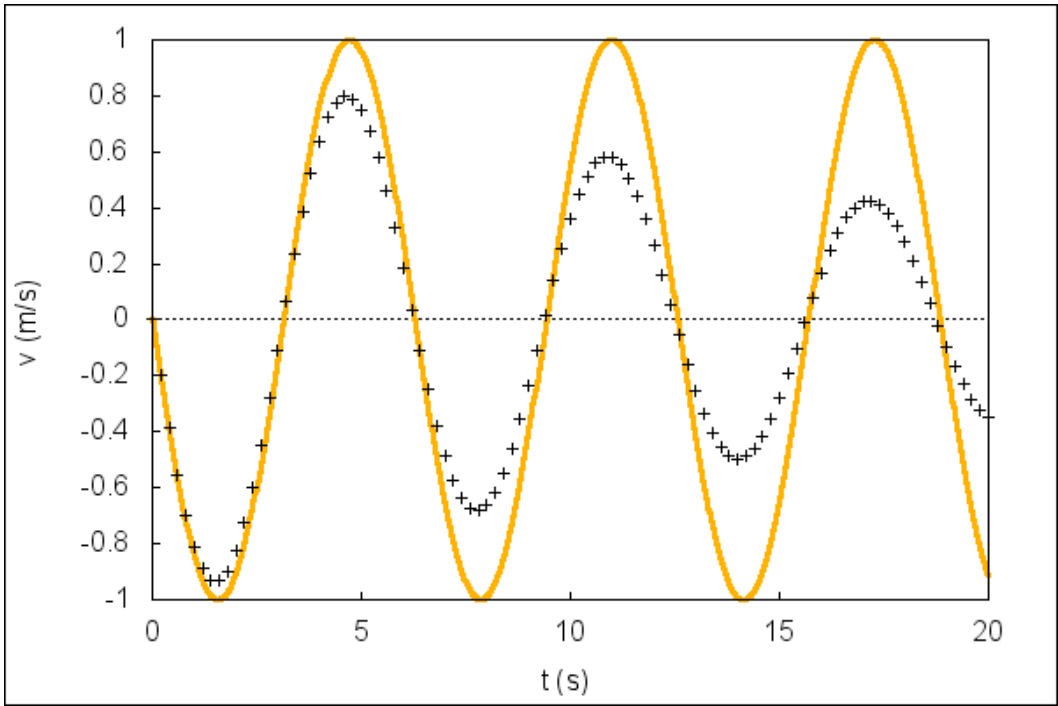
Euler-Cromer algorithm

$$
\begin{aligned}
v_{n+1} &= v_n - \frac{k}{m} x_n \Delta t - c v_n \Delta t \\
x_{n+1} &= x_n + v_{n+1} \Delta t \\
&= x_n + v_n \Delta t - \frac{k}{m} x_n \Delta t^2 - c v_n \Delta t^2
\end{aligned}
$$

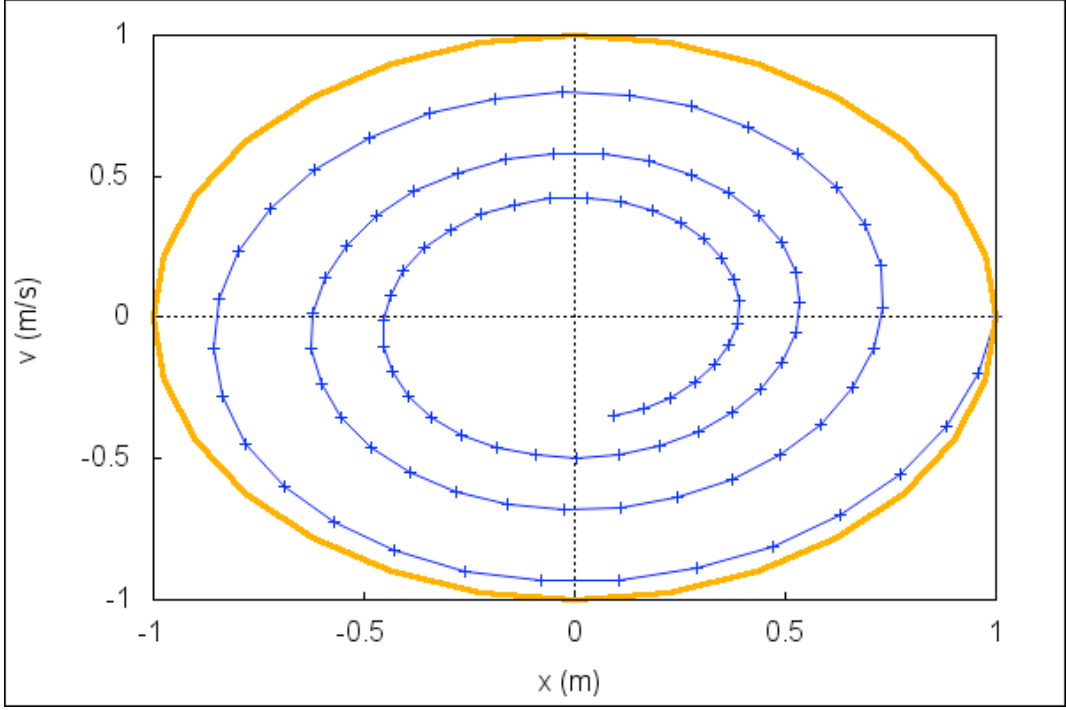
$$
|J| = 1 - c\Delta t
$$

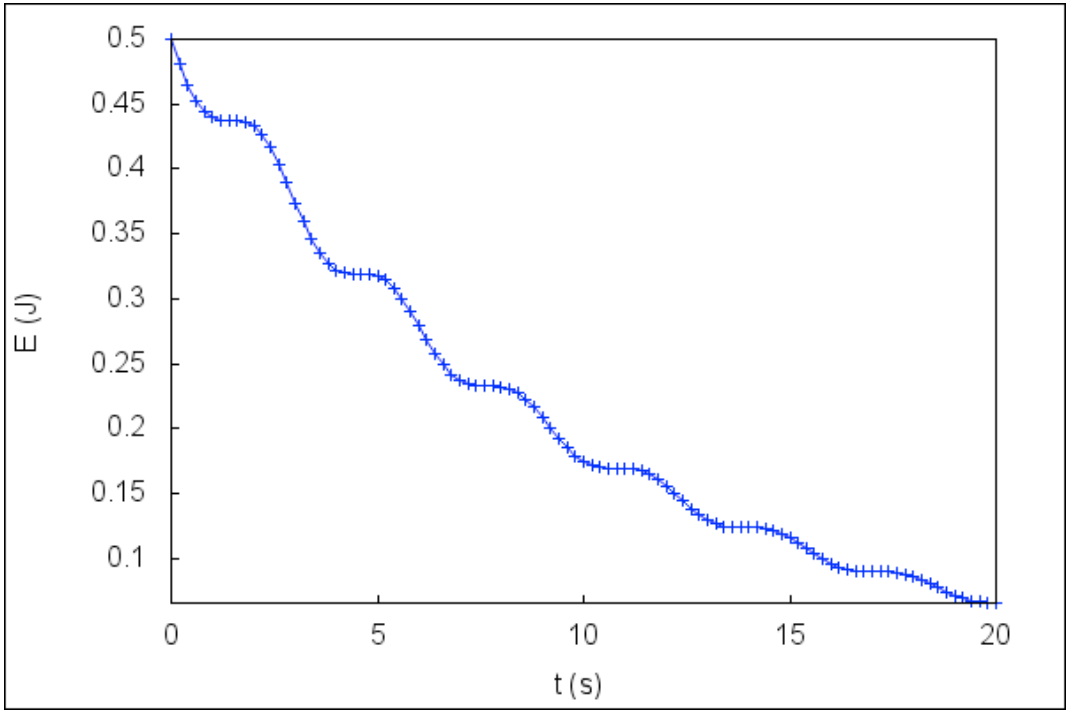
# Position and Velocity Plots



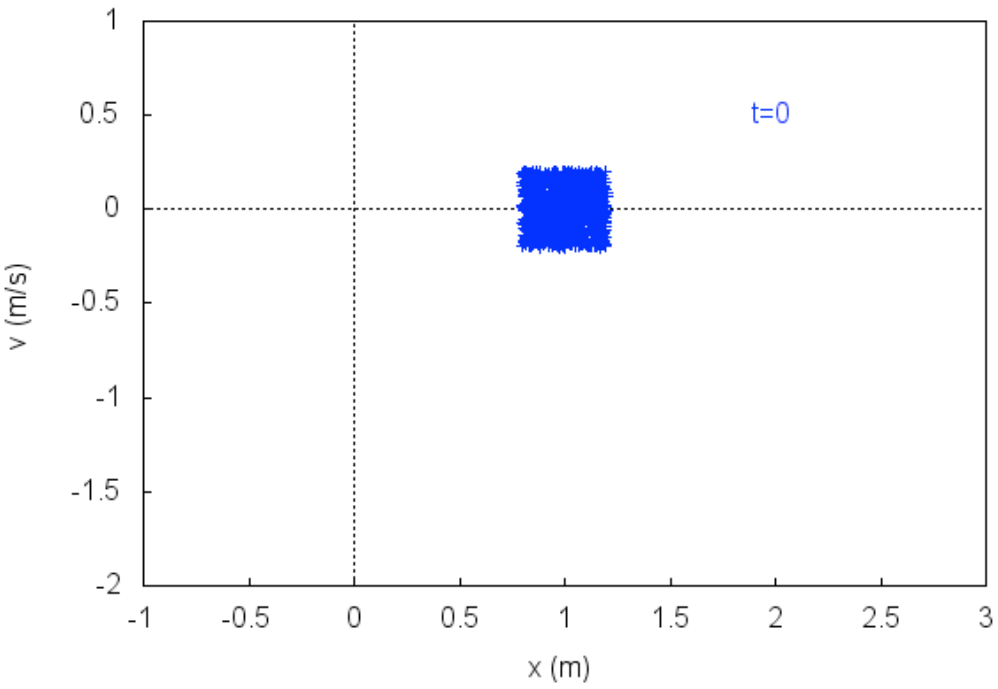



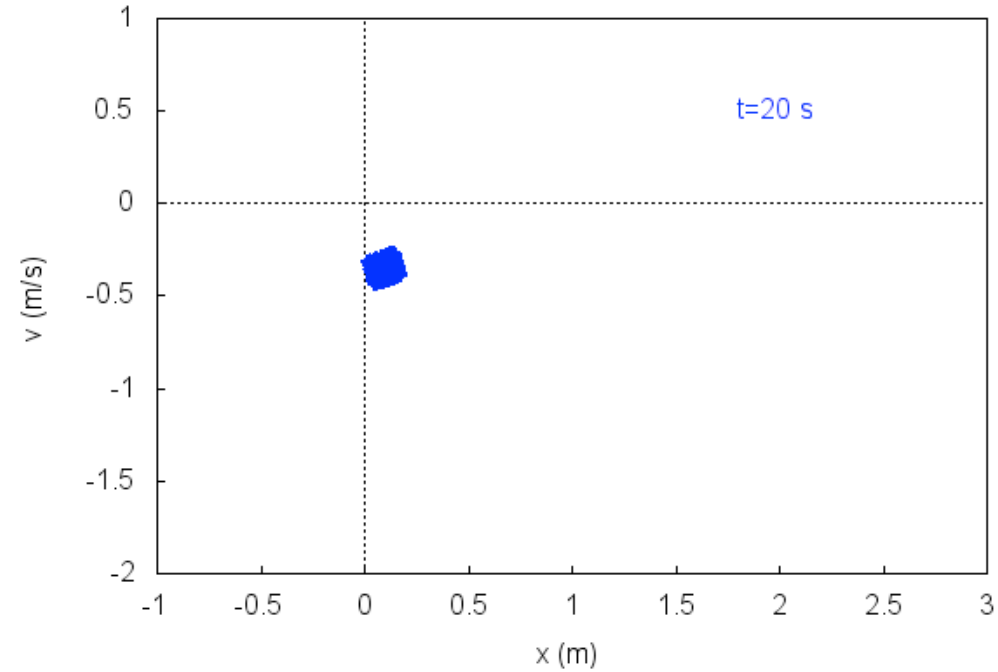
# Phase Space Path and Energy Plot



# Evolution of Ensemble in Phase Space



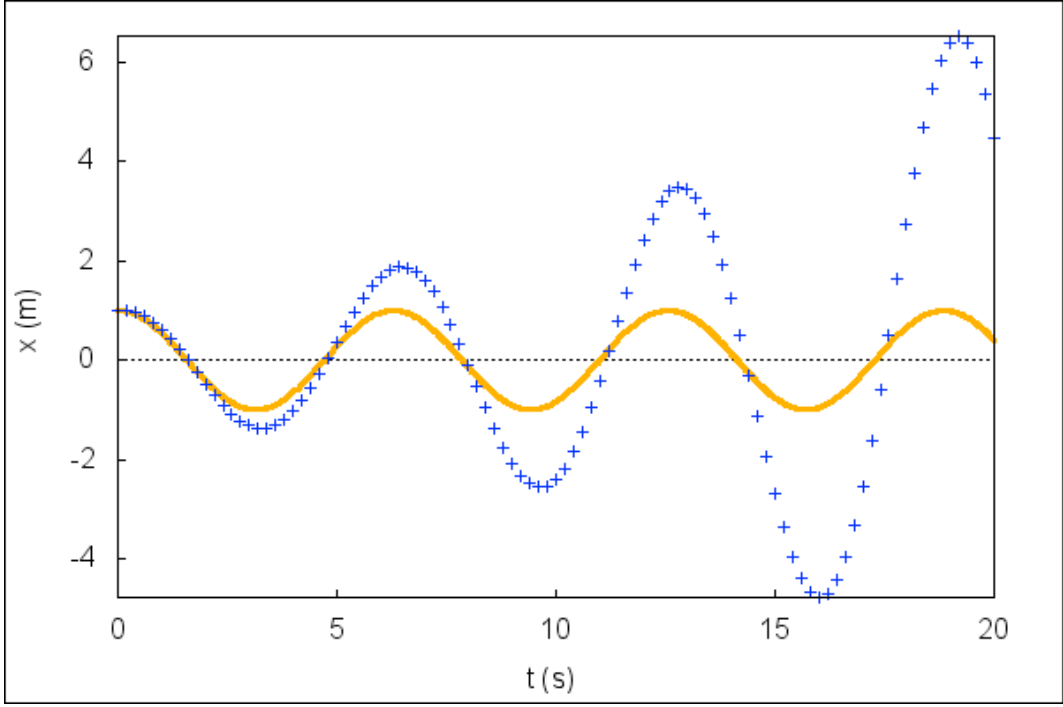
# Harmonic Oscillator no damping

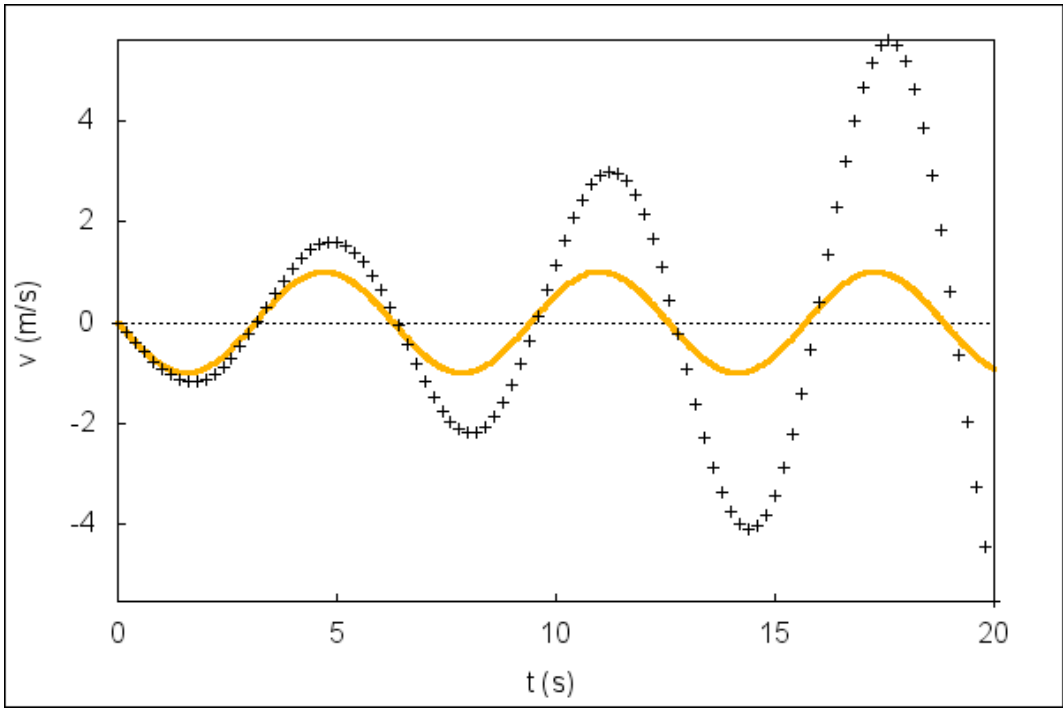
Euler algorithm

$$
\begin{aligned}
x_{n+1} &= x_n + v_n \Delta t \\
v_{n+1} &= v_n - \frac{k}{m} x_n \Delta t
\end{aligned}
$$

$$
|J| = 1 + k\Delta t^2/m
$$

# Position and Velocity Plots



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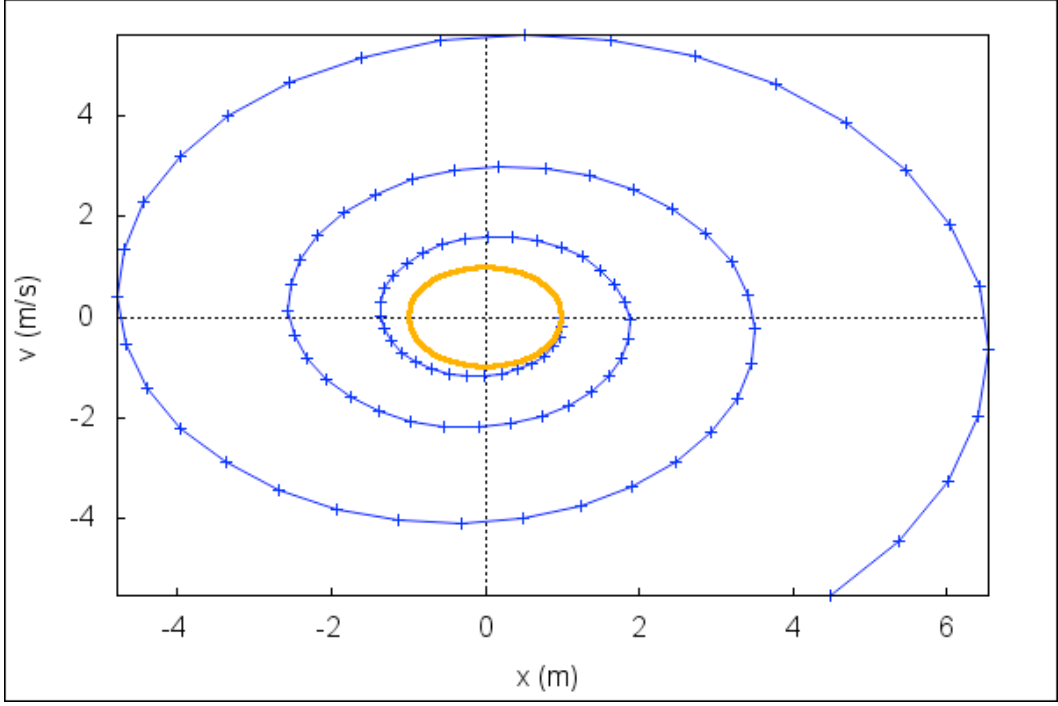


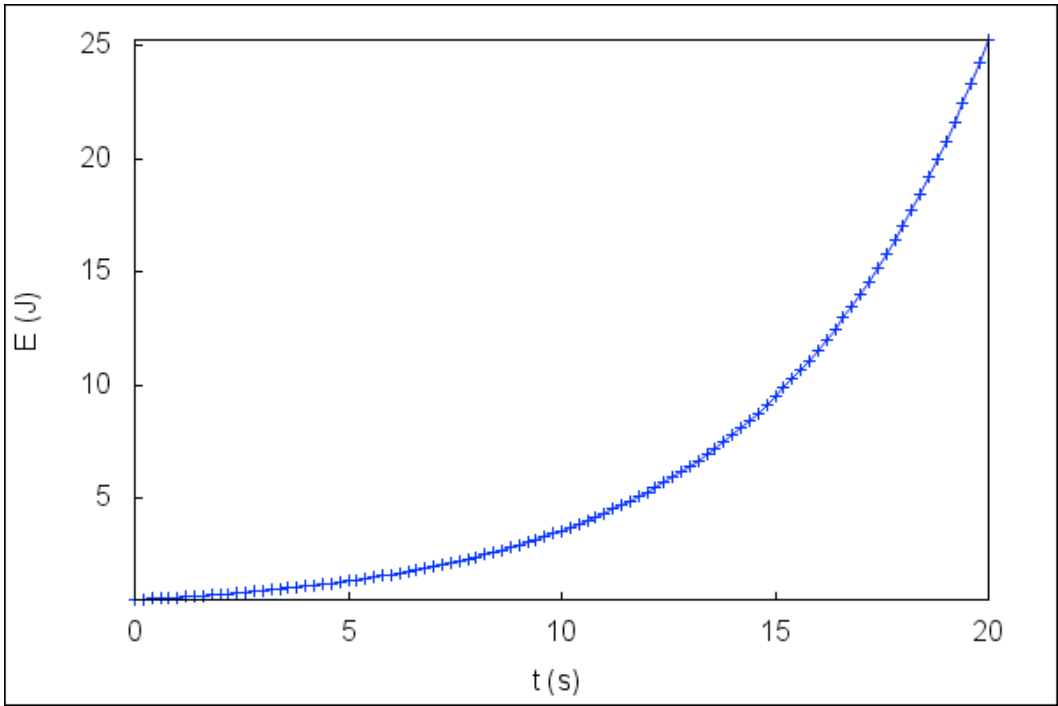

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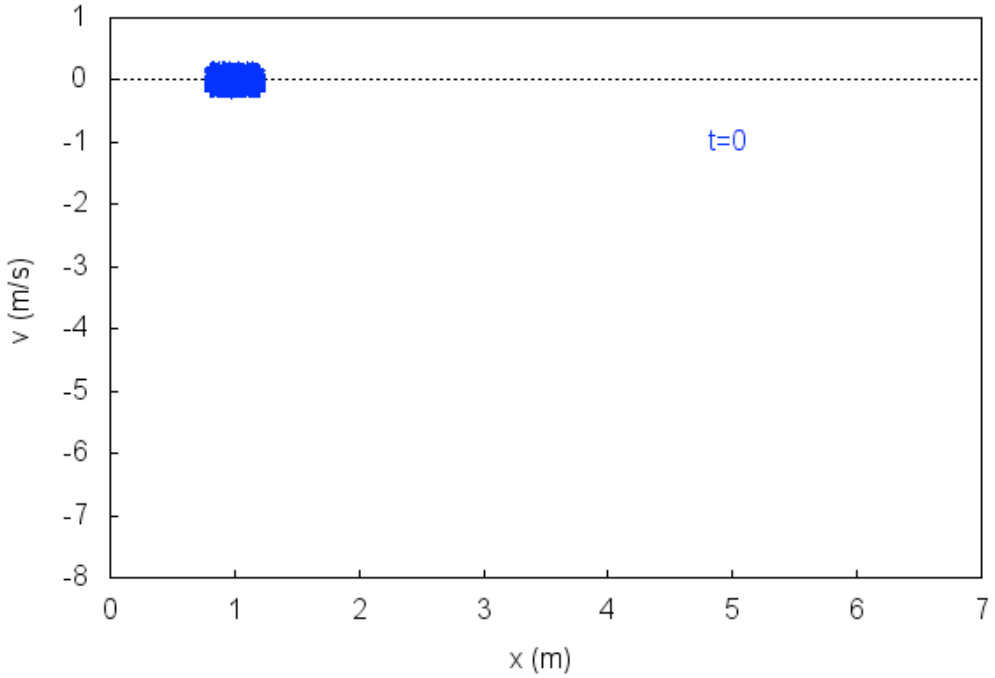
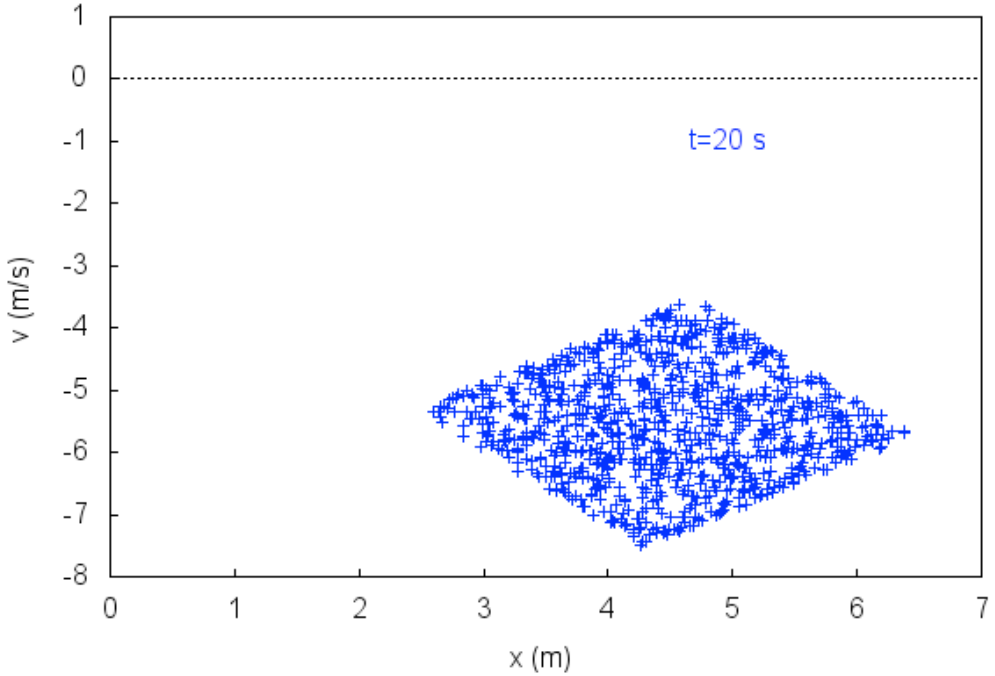


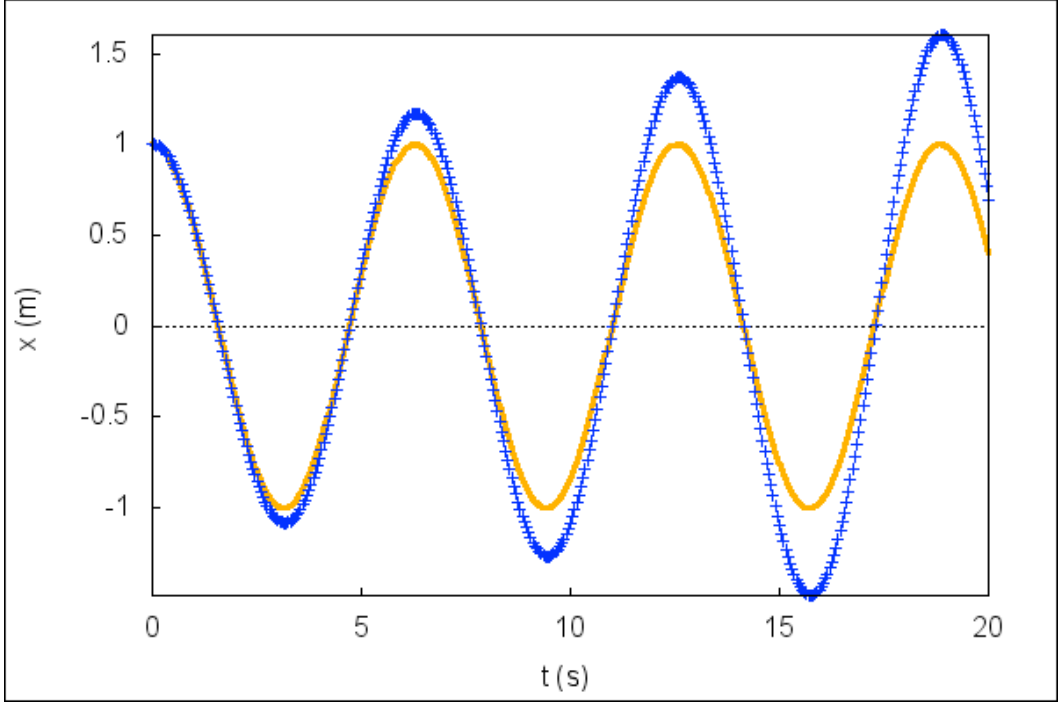
# Harmonic Oscillator
# no damping

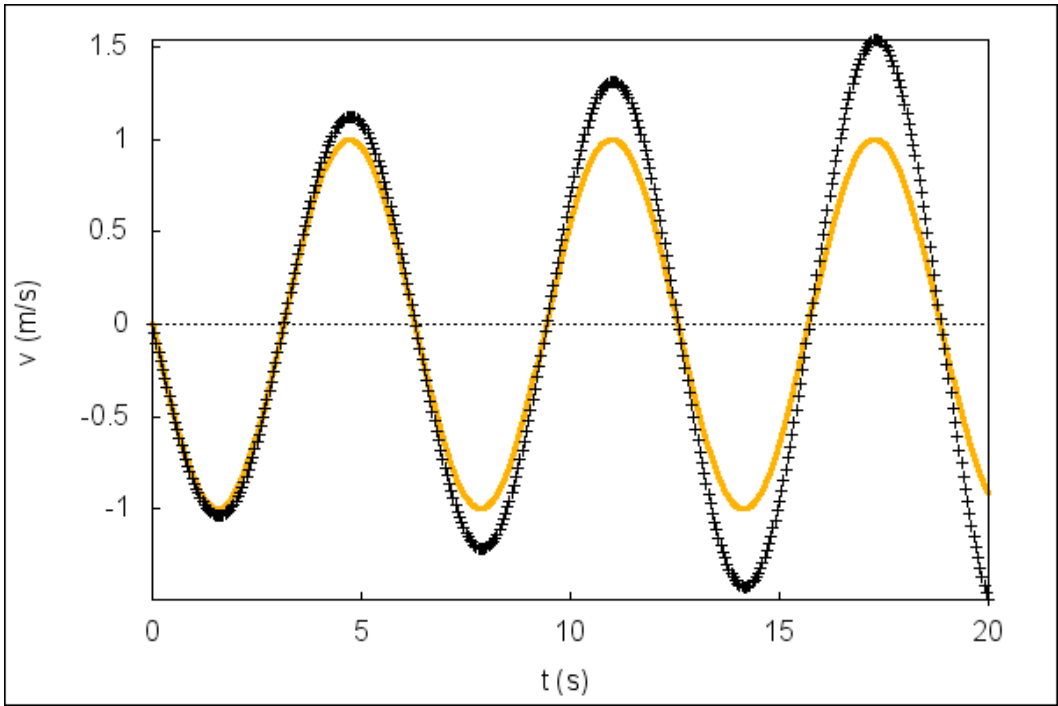
Euler algorithm, shorter time step

$$
\begin{aligned}
x_{n+1} &= x_n + v_n \Delta t \\
v_{n+1} &= v_n - \frac{k}{m} x_n \Delta t
\end{aligned}
$$

$$
|J| = 1 + k\Delta t^2/m
$$

# Position and Velocity Plots



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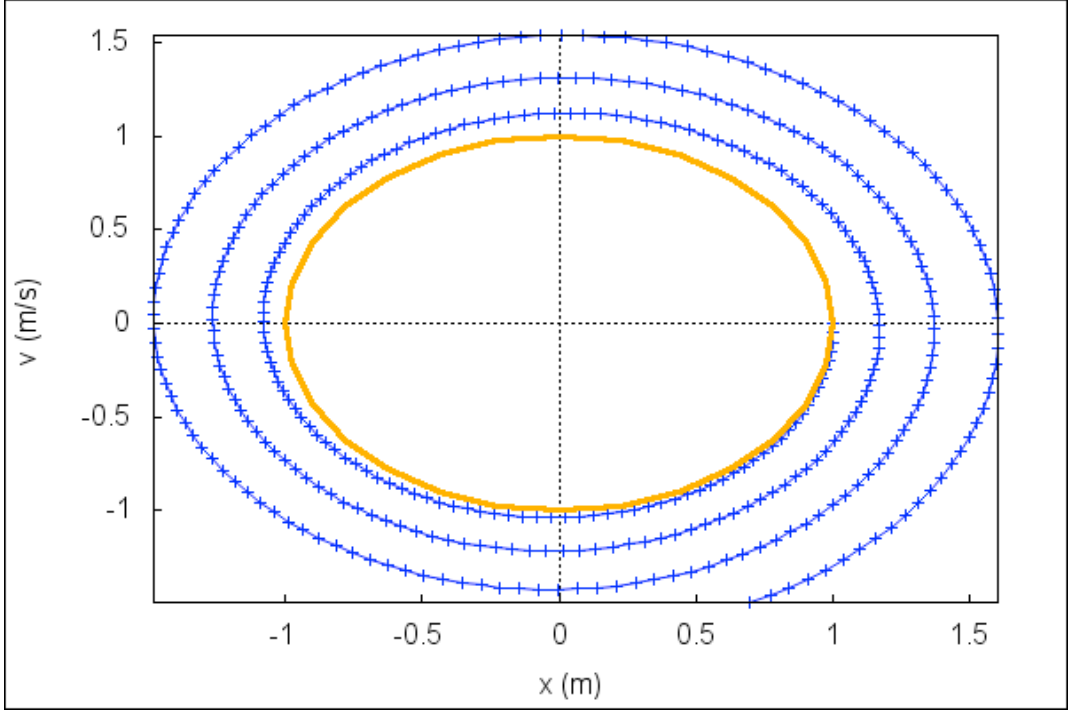


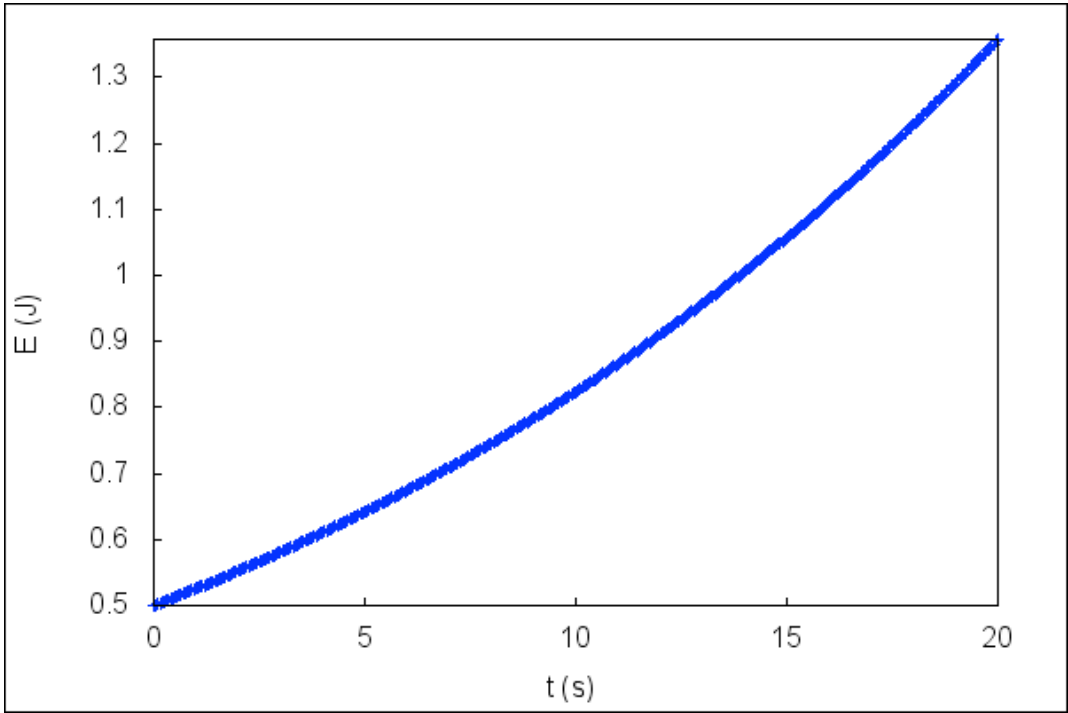

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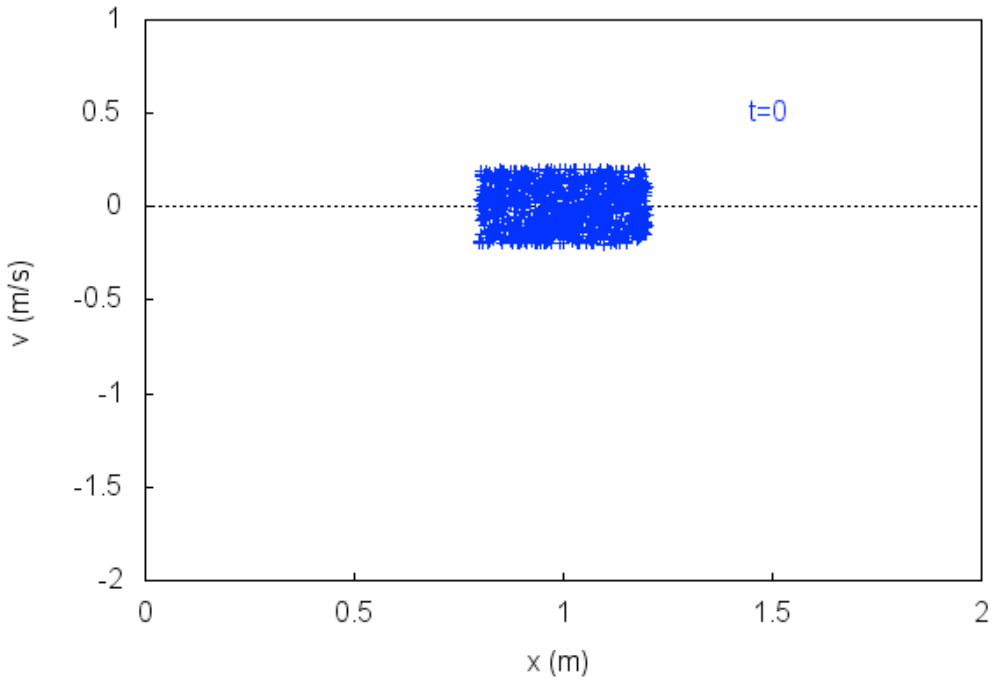
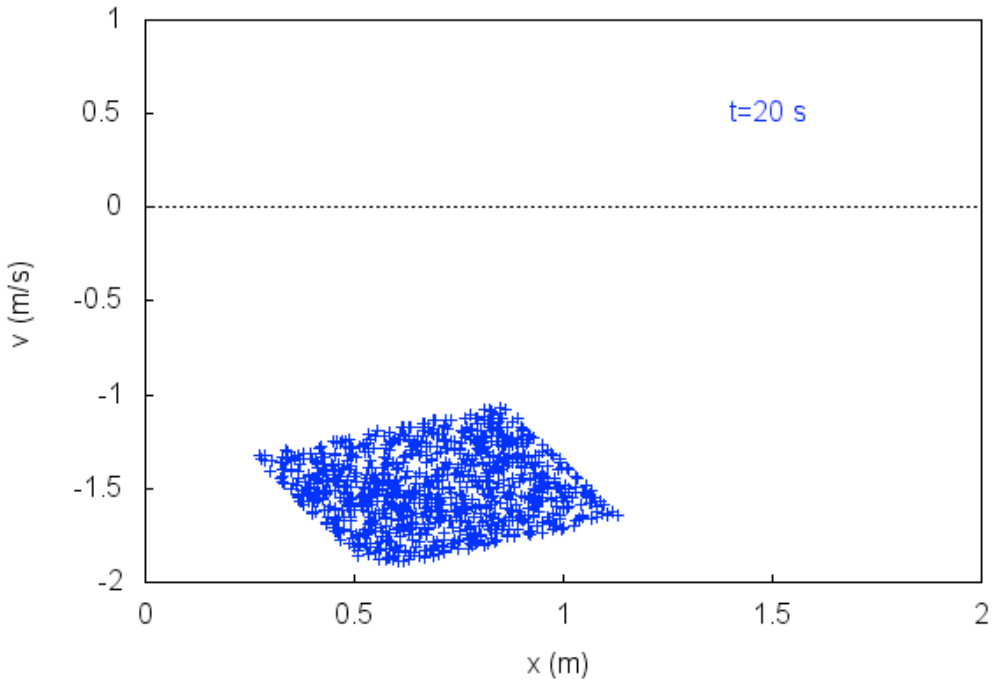
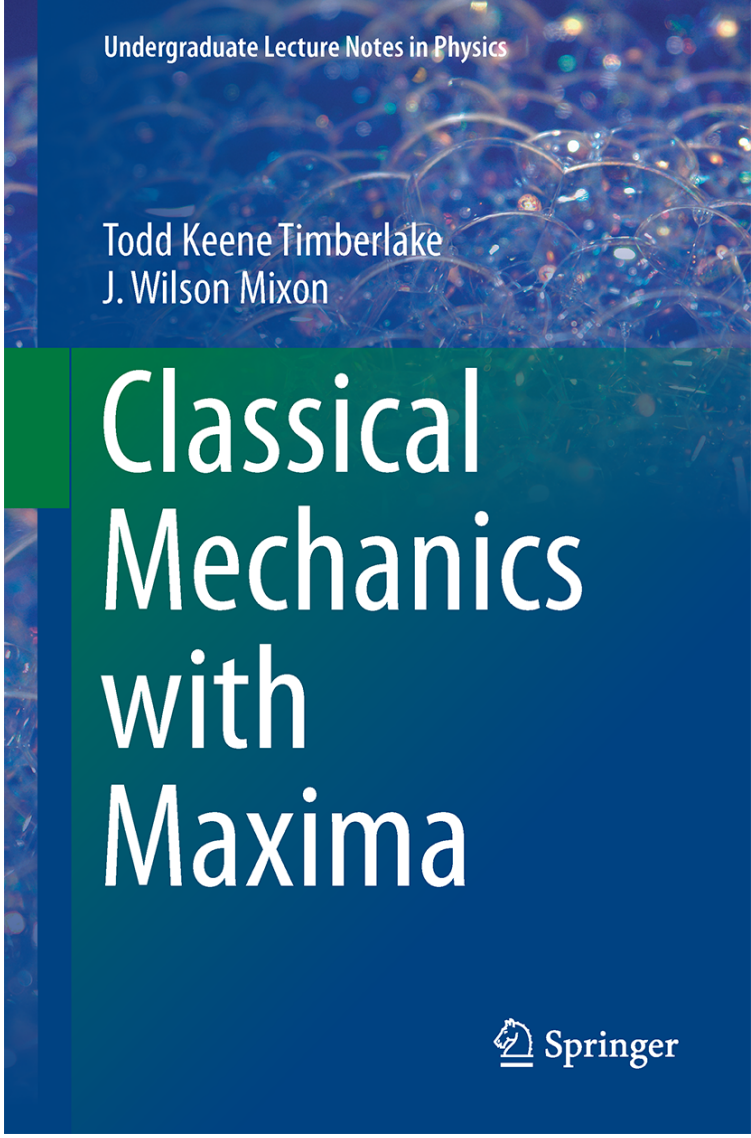


# References

- Timberlake and Mixon, *Classical Mechanics with Maxima* (Springer 2015).
- Timberlake and Hasbun, "Computation in classical mechanics", *AJP* **76** 334.
- Maxima notebooks available at:




sites.berry.edu/ttimberlake/teaching/cm_maxima/

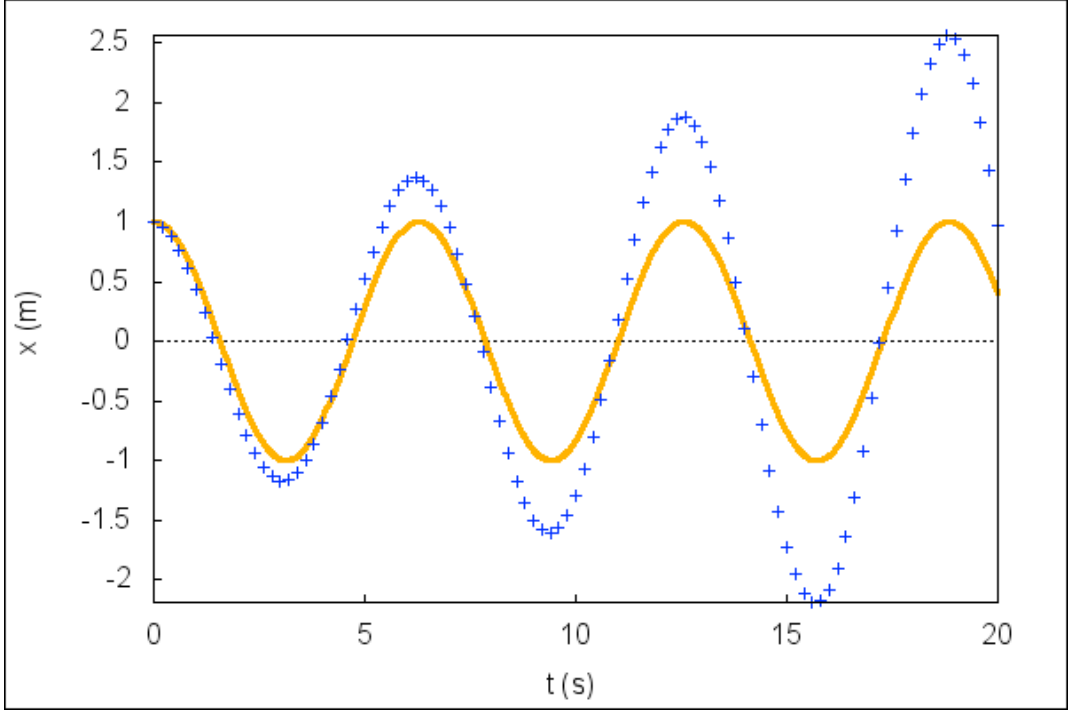
# Extra: Harmonic Oscillator linear anti-damping

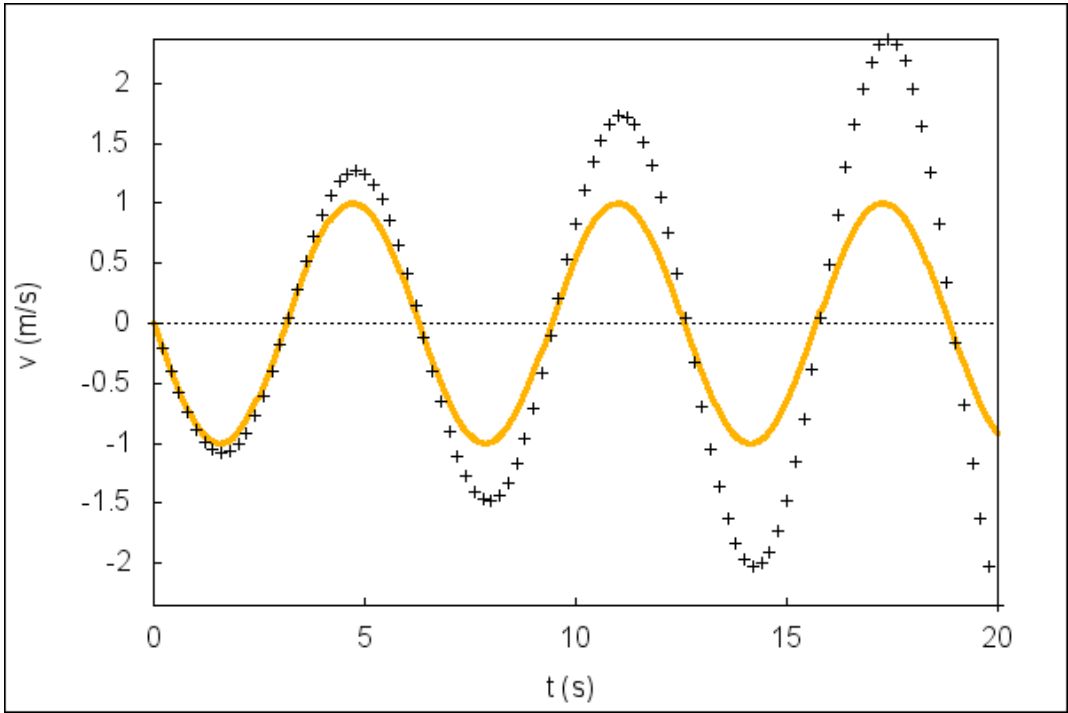
Euler-Cromer algorithm

$$
\begin{aligned}
v_{n+1} &= v_n - \frac{k}{m} x_n \Delta t + c v_n \Delta t \\
x_{n+1} &= x_n + v_{n+1} \Delta t \\
&= x_n + v_n \Delta t - \frac{k}{m} x_n \Delta t^2 + c v_n \Delta t^2
\end{aligned}
$$

$$
|J| = 1 + c\Delta t
$$

# Position and Velocity Plots



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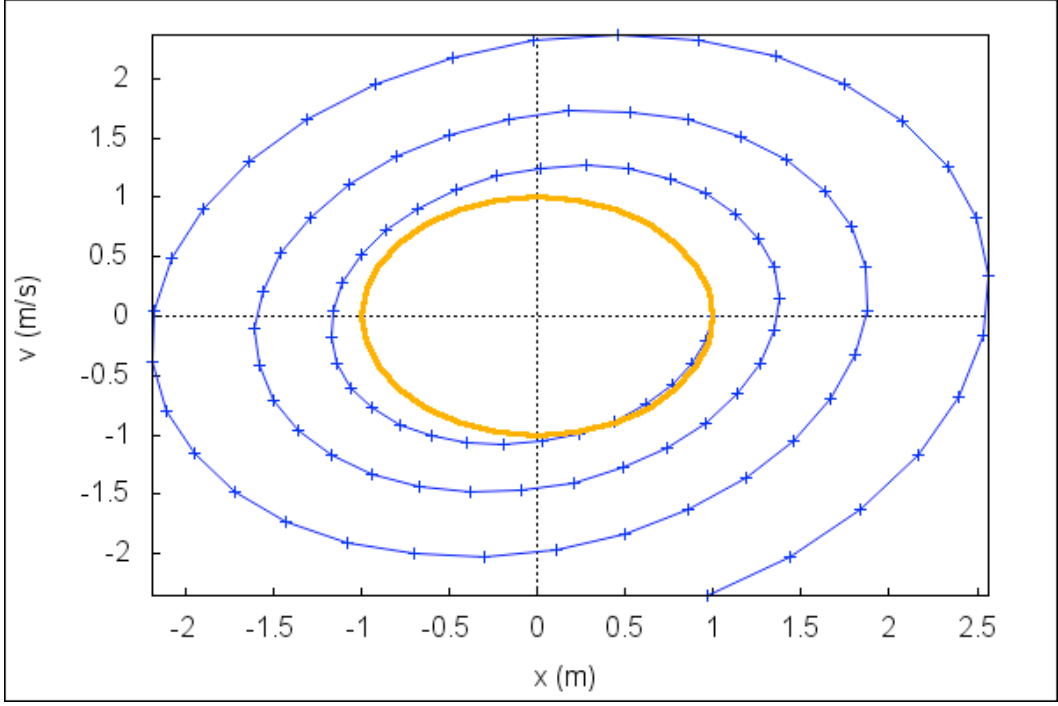


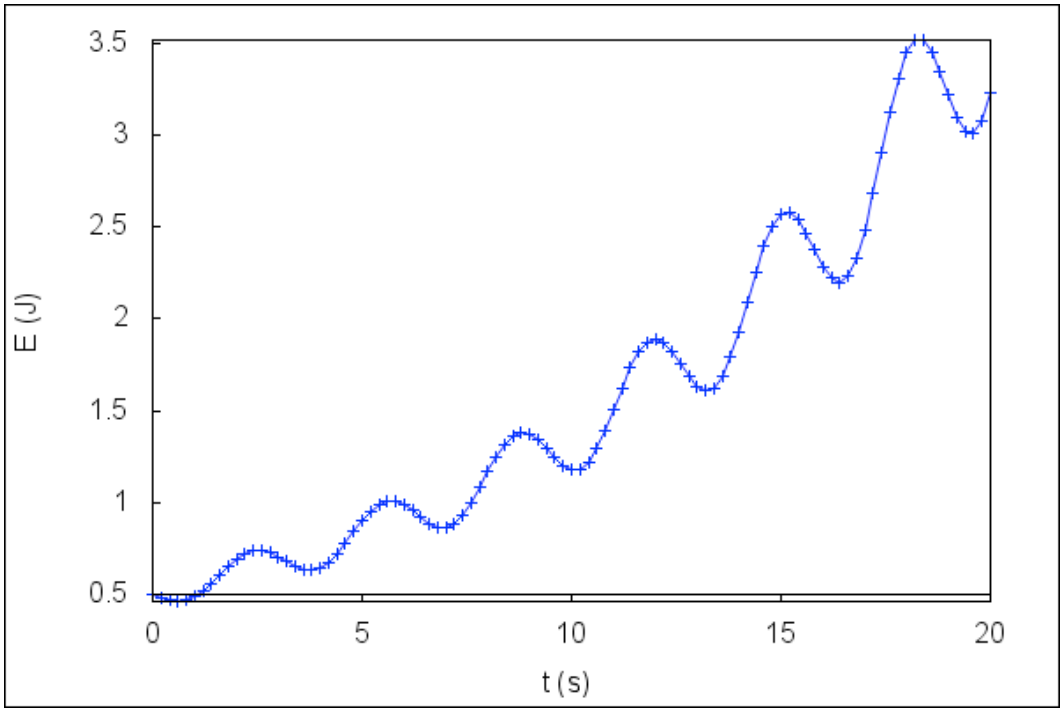

# Evolution of Ensemble in Phase Space